Homogenization problems from shallow water theory

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This note is devoted to the effect of topography on geophysical flows. We consider two models derived from shallow water theory: the quasigeostrophic equation and the lake equation. Small scale variations of topography appear in these models through a periodic function, of small wavelength ε. The asymptotic limit as ε goes to zero reveals homogenization problems in which the cell and averaged equations are both nonlinear. In the spirit of article [1], we derive rigorously the limit systems, through the notion of two-scale convergence.

1 Introduction and formal derivation

The inviscid shallow water equations read, in a bounded domain Ω:

$$\begin{cases} \partial_t h + \operatorname{div}(hu) = 0, \quad t > 0, \quad x = (x_1, x_2) \in \Omega, \\ \partial_t u + u \cdot \nabla u + \dfrac{u^\perp}{\mathrm{Ro}} = \dfrac{1}{\mathrm{Fr}^2} \nabla (h + h_b), \quad t > 0, \quad x \in \Omega, \\ u \cdot n|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \end{cases}$$

The unknowns are the height of water $h = h(t,x)$ and the horizontal velocity field $u = u(t,x)$. The function $h_b = h_b(x)$ describes the bottom topography. The positive parameters Ro and Fr are called the Rossby and Froude numbers, they penalize the Coriolis force $u^\perp$ and the pressure term $\nabla (h + h_b)$ respectively. We refer to [4] for all details and possible extensions.

In many questions of geophysical concern, at least one of the parameters Ro or Fr is very small, which leads asymptotically to reduced models. A standard one is the "quasigeostrophic equation", obtained in the scaling $\mathrm{Ro} \approx \mathrm{Fr} \approx ||h_b|| \ll 1$. It reads in its simplest form, see textbook [11]:

$$\begin{cases} (\partial_t + u \cdot \nabla) (\Delta\psi - \psi + \eta_b) = 0, \quad t > 0, \quad x \in \Omega, \\ u = \nabla^\perp \psi, \quad u \cdot n|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{1}$$

where ψ is the streamfunction associated to the velocity u, and η_b the (rescaled) bottom topography. Another classical one is the "lake equation", corresponding to the asymptotics $\mathrm{Ro} \gg 1$, $\|h_b\| \approx 1$, $Fr \ll 1$. It leads to, *c.f.* [8]:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad t > 0, \quad x \in \Omega, \\ \operatorname{div}(\eta_b u) = 0, \quad t > 0, \quad x \in \Omega, \\ u \cdot n|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{2}$$

Note that for non-varying bottom $\eta_b = 1$, (2) resumes to incompressible Euler equations.

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The present work is part of an attempt to describe mathematically the impact of variations of relief on geophysical models. It follows former studies [6, 3] by the authors on roughness-induced effects. These studies focused on boundary layer problems (Ekman layers for rotating fluids, Munk layers for the viscous quasigeostrophic equation). We consider here the effect of rapid variations of the topography on systems (1) and (2). We assume in both cases that

$$\eta_b(x) \;=\; \eta\left(x, x/\varepsilon\right), \quad \eta \;:=\; \eta(x,y) \;\in \mathcal{C}^0\left(\overline{\Omega}\times\mathbb{T}^2\right), \quad 0<\varepsilon\le 1,$$

where $\mathbb{T}^2 \;:=\; (\mathbb{R}/\mathbb{Z})^2$ models periodic oscillations of the bottom. We also suppose that η is bounded from above and below by positive constants. To unify notations, we denote η_ε instead of η_b. We will limit ourselves to the weakly nonlinear regime for which $\psi = O(\varepsilon)$, $u = O(\varepsilon)$. Through the change of variables $\psi = \varepsilon\Psi$, $u = \varepsilon v$, previous systems become

$$\begin{cases} (\partial_t \;+\; \varepsilon v\,\cdot\,\nabla)\;(\Delta\Psi \;-\; \Psi) \;+\; v\,\cdot\,\nabla\eta_\varepsilon \;=\; 0, \quad t>0, \quad x\in\Omega, \\ v = \nabla^\perp\Psi, \quad v\cdot n|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0^\varepsilon, \end{cases} \tag{3}$$

respectively

$$\begin{cases} \partial_t v \;+\; \varepsilon v\cdot\nabla v \;+\; \nabla p \;=\; 0, \quad t>0, \quad x\in\Omega, \\ \operatorname{div}(\eta_\varepsilon v) = 0, \quad t>0, \quad x\in\Omega, \\ v\cdot n|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0^\varepsilon. \end{cases} \tag{4}$$

We wish to understand the influence of η_ε at large scale, as its typical small wavelength ε goes to zero. This yields homogenization problems, that can be tackled at a formal level by double-scale expansions:

$$\begin{aligned} \Psi^\varepsilon \;&\sim\; \Psi^0\left(t,x\right) \;+\; \varepsilon\Psi^1\left(t,x,x/\varepsilon\right) \;+\; \varepsilon^2\Psi^2\left(t,x,x/\varepsilon\right) \;+\; \ldots \\ v^\varepsilon \;&\sim\; v^0\left(t,x,x/\varepsilon\right) \;+\; \varepsilon v^1\left(t,x,x/\varepsilon\right) \;+\; \ldots \end{aligned}$$

In the quasigeostrophic model, this expansion yields easily:

$$\begin{cases} \left(\partial_t \;+ v^0\cdot\nabla_y\right)\,\Delta_y\Psi^1 + v^0\cdot\nabla_y\eta \;=\; 0, \\ \partial_t\left(\Delta_x\Psi^0 - \Psi^0\right) + \overline{v^0\cdot\nabla_x\eta} \;+\; \overline{\nabla_x^\perp\Psi^1\cdot\nabla_y\eta} \;=\; 0, \\ v^0 = \nabla_x\Psi^0 + \nabla_y\Psi^1, \quad \overline{v}^0\cdot n|_{\partial\Omega} = 0, \quad \overline{v}^0|_{t=0} = v_0, \end{cases} \tag{5}$$

with $\overline{f} := \int_{\mathbb{T}^2} f(\cdot,y)dy$. These cell and averaged equations form a coupled nonlinear system. The linearized version of this system has been studied from a physical viewpoint in [7].

In the lake model, the Ansatz leads to

$$\begin{cases} \partial_t v^0 \;+\; v^0\cdot\nabla_y v^0 \;+\; \nabla_x p^0 + \nabla_y p^1 \;=\; 0, \\ \operatorname{div}_y\left(\eta\, v^0\right) = 0, \quad \operatorname{div}_x\left(\overline{\eta\, v^0}\right) = 0, \\ \overline{v}^0\cdot n|_{\partial\Omega} = 0, \quad \overline{v}^0|_{t=0} = v_0. \end{cases} \tag{6}$$

The present paper deals with the convergence of (3) toward (5), resp. (4) toward (6). The convergence result involves the notion of two-scale convergence introduced by Nguetseng [14], and developed by Allaire [13]. We give here an enlarged definition, that accounts for time variable and various Lebesgue spaces. Classical properties of two-scale convergence extend to this framework (see [13]).

Definition 1 *Let Ω a bounded domain of $\mathbb{R}^2$. Let $v^\varepsilon = v^\varepsilon(t,x)$ a sequence of functions in $L^p\left(\mathbb{R}_+;\, L^q(\Omega)\right)$, with $1 < p,q \le \infty$, $(p,q) \neq (\infty,\infty)$ (respectively in $L^\infty\left(\mathbb{R}_+ \times \Omega\right)$). Let $v \in L^p\left(\mathbb{R}_+;\, L^q\left(\Omega \times \mathbb{T}^2\right)\right)$ (respectively in $L^\infty\left(\mathbb{R}^+ \times \Omega \times \mathbb{T}^2\right)$). We say that v^ε two-scale converges to v if, for all $w \in L^{p'}\left(\mathbb{R}_+;\, C^0(\overline{\Omega} \times \mathbb{T}^2)\right)$,*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}_+} \int_{\Omega} v^\varepsilon(t,x) w\left(t,x,x/\varepsilon\right) dx dt \; = \; \int_{\mathbb{R}_+} \int_{\Omega \times \mathbb{T}^2} v(t,x,y)\, w(t,x,y)\, dx\, dy\, dt.$$

For functions independent of t, $p = \infty$ and $q = 2$, one recovers the two-scale convergence on Ω. We state the main result:

Theorem 2 *Let $q_0 > 2$, v_0^ε bounded in $L^{q_0}(\Omega)$, satisfying $div\, v_0^\varepsilon = 0$, $v_0^\varepsilon \cdot n|_{\partial\Omega} = 0$, $\varepsilon \operatorname{curl} v_0^\varepsilon$ bounded in $L^\infty(\Omega)$. Assume moreover that v_0^ε two-scale converges to v_0 on Ω, with*

$$\|v_0^\varepsilon\|_{L^2(\Omega)} \xrightarrow[\varepsilon \to 0]{} \|v_0\|_{L^2(\Omega \times \mathbb{T}^2)}.$$

Then, up to extract a subsequence, the solution v^ε of (3), resp. (4) two-scale converges to a solution v^0 of (5), resp. (6).

2 Justification

The proof of the convergence result follows the strategy used in [1], devoted to Euler equations in a porous medium: *one tries to obtain integral inequalities on the defect measure of v^ε.* Notice that (3) and (5) can be reformulated in an Eulerian framework, namely

$$(\partial_t + \varepsilon v \cdot \nabla)\left(v - \Delta^{-1} v\right) \; + \; \eta_\varepsilon\, v^\perp \; + \; \nabla p \; = \; 0, \quad \operatorname{div} v = 0,$$

and

$$\begin{cases} \partial_t \left(v^0 - \Delta_x^{-1}\, \overline{v}^0\right) \; + \; v^0 \cdot \nabla_y v^0 \; + \; \eta \, \left(v^0\right)^\perp \; + \; \nabla_x p^0 + \nabla_y p^1 \; = \; 0, \\ \operatorname{div}_y v^0 = 0, \quad \operatorname{div}_x \overline{v}^0 = 0, \end{cases}$$

where Δ^{-1} is the inverse of the Laplacian with homogeneous Dirichlet boundary condition. Hence, we will focus on the lake equation, the quasigeostrophic case being simpler and in the same line. Note that at fixed $\varepsilon > 0$, by Yudovitch type theorem (see [8]), the assumptions on the initial data ensure the existence and uniqueness of a global in time solution v^ε. More precisely, we have

$$v^\varepsilon \;\text{ bounded in }\; L^\infty\left(\mathbb{R}_+;\, L^2(\Omega, \eta_\varepsilon dx)\right) = L^\infty\left(\mathbb{R}_+;\, L^2(\Omega)\right),$$

and using the transport equation $\partial_t \omega^\varepsilon + \varepsilon v^\varepsilon \cdot \nabla \omega^\varepsilon = 0$, $\omega^\varepsilon = \operatorname{curl}(v^\varepsilon)/\eta_\varepsilon$, we deduce

$$\varepsilon \operatorname{curl}(v^\varepsilon) \;\text{ bounded in }\; L^\infty\left(\mathbb{R}_+ \times \Omega\right).$$

We first remind a classical result from elliptic theory, due to Meyers [9]. *Let Ω a bounded domain of $\mathbb{R}^2$, $a \in L^\infty(\Omega)$, and $f \in L^{q_0}(\Omega)$ for some $q_0 > 2$, $\int_{\partial\Omega} f \cdot n = 0$. Let $\phi \in H^1(\Omega)$ the solution of*

$$div\,(a \nabla \phi) = div\, f \;\; in\; \Omega, \qquad \int_\Omega \phi = 0, \quad \partial_n \phi|_{\partial\Omega} = 0.$$

There exists $2 < q_m = q_m(\|a\|_{L^\infty}, \Omega) \le q_0$, *such that for all* $2 \le q < q_m$, $\phi \in W^{1,q}(\Omega)$ *with*

$$\|\phi\|_{W^{1,q}} \;\le\; C\,\|f\|_{L^q}, \quad C = C\,(q, \|a\|_{L^\infty}, \Omega)\,. \tag{7}$$

The original result of Meyers deals with Dirichlet condition, but extends easily to Neumann condition. Article [9] also contains examples for which $q_m < q_0$. It thus differs greatly from the case of smooth coefficients (*i.e.* a smooth), for which estimate (7) holds for all q, but with a constant C involving derivatives of a. We deduce from this result that the solution v^ε of (4) satisfies, for all $2 < q < q_m \le q_0$,

$$\|v^\varepsilon\|_{L^q} \;\le\; C\,\|\mathbb{P}v^\varepsilon\|_{L^q}, \quad C = C\,(\sup\|\eta_\varepsilon\|_{L^\infty}, \Omega)\,. \tag{8}$$

Indeed the Helmoltz decomposition

$$v^\varepsilon = \mathbb{P}v^\varepsilon + \nabla\phi^\varepsilon, \quad \int_\Omega \phi^\varepsilon = 0,$$

together with $\mathrm{div}\,(\eta_\varepsilon v^\varepsilon) = 0$ yields

$$\mathrm{div}\;(\eta_\varepsilon \nabla\phi^\varepsilon) = -\mathrm{div}\,(\eta_\varepsilon \mathbb{P}v^\varepsilon), \quad \int_\Omega \phi^\varepsilon = 0, \quad \partial_n\phi^\varepsilon|_{\partial\Omega} = 0.$$

and (7) provides the claimed estimate.

We now rewrite equation (4a) as

$$\partial_t v \;+\; \varepsilon\;(\mathrm{curl}\,v)\;v^\perp \;+\; \nabla\left(p^\varepsilon + \frac{\varepsilon|v|^2}{2}\right) = 0, \tag{9}$$

which yields

$$\partial_t\,\mathbb{P}v + \mathbb{P}\left(\varepsilon\,\mathrm{curl}\,v\,v^\perp\right) = 0.$$

Multiplying by $|v|^{q-2}v$, for some $2 < q < q_m$, we deduce

$$\|\mathbb{P}v^\varepsilon(t)\|^q_{L^q} \;\le\; C\left(\|v^\varepsilon_0\|^q_{L^q} + \int_0^t \|v^\varepsilon(s)\|^q_{L^q}\,ds\right).$$

By (8), we get that v^ε is bounded in $L^\infty(\mathbb{R}_+, L^q(\Omega))$. It follows easily that $\partial_t v^\varepsilon$ and $\nabla\left(p^\varepsilon + \varepsilon|v^\varepsilon|^2/2\right)$ are bounded in $L^\infty(\mathbb{R}_+, L^q(\Omega))$.

Up to consider subsequences, these bounds imply that v^ε two-scale converges to some $v^0 \in L^\infty\left(\mathbb{R}_+;\, L^q(\Omega\times\mathbb{T}^2)\right)$. By classical properties of two scale convergence (see [13] for all details), $\varepsilon\,\mathrm{curl}\,v^\varepsilon$ two scale converges to $\mathrm{curl}_y\,v^0 \in L^\infty\left(\mathbb{R}_+\times\Omega\times\mathbb{T}^2\right)$. We also get $\mathrm{div}_y\,(\eta\,v^0) = 0$, $\mathrm{div}_x\,\overline{\eta\,v^0} = 0$, and $\overline{v}^0\cdot n|_{\partial\Omega} = 0$. Besides, $q^\varepsilon = p^\varepsilon + \varepsilon|v^\varepsilon|^2/2$ converges weakly-star in $L^\infty(\mathbb{R}_+;\, W^{1,q}(\Omega))$ to $q = q(t,x)$, and ∇q^ε two-scale converges to $\nabla_x q + \nabla_y p$ for some $p = p(t,x,y) \in L^\infty\left(\mathbb{R}_+;\, L^q\left(\Omega\times\mathbb{T}^2\right)\right)$.

Finally, we introduce the two-scale defect measure α, resp. β such that $|v^\varepsilon|^2$ two-scale converges to $|v^0|^2 + \alpha$, resp. $v^\varepsilon\otimes v^\varepsilon$ two-scale converges to $v^0\otimes v^0 + \beta$. Note that $\alpha, \beta_{i,j} \in L^\infty\left(\mathbb{R}_+;\, L^{q/2}\left(\Omega\times\mathbb{T}^2\right)\right)$. We stress that for any quantity f^ε that two-scale converges to f, $\eta_\varepsilon\,f^\varepsilon$ two-scale converges to ηf. We multiply (4a) by η_ε and take the limit $\varepsilon\to 0$. We obtain

$$\eta\partial_t v^0 + \mathrm{div}_y\;\left(\eta v^0\otimes v^0 + \eta\beta\right) + \eta\nabla_x q + \eta\nabla_y\tilde{p} \;=\; 0,$$

where $\tilde{p} = p - \alpha/2 \in L^\infty\left(\mathbb{R}_+;\, L^{q/2}\left(\Omega \times \mathbb{T}^2\right)\right)$. We multiply by v^0 and integrate over $\mathbb{T}^2$:

$$\partial_t \, \overline{\frac{\eta\,|v^0|^2}{2}} \; + \; \overline{\eta\beta : \nabla_y v^0} \; = \; -\mathrm{div}_x \left(\overline{\eta\, v^0}\, q \right). \tag{10}$$

Then, we deduce from (9), (10):

$$\partial_t \frac{\eta_\varepsilon |v^\varepsilon|^2}{2} = -\mathrm{div}\, \left(\eta_\varepsilon\, v^\varepsilon\, q^\varepsilon\right).$$

We remind that $v^\varepsilon, \partial_t v^\varepsilon$ are bounded in $L^\infty\left(\mathbb{R}_+;\, L^q(\Omega \times \mathbb{T}^2)\right)$. By Aubin-Lions lemma, we deduce that

$$\left(v^\varepsilon\right)_\varepsilon \;\text{ is strongly compact in }\; L^\infty\left(W^{-1,q'}(\Omega)\right).$$

As q^ε converges weakly to q in $L^2\left(\mathbb{R}_+;\, W^{1,q}(\Omega)\right)$, we may pass to the limit in previous equation, and derive

$$\partial_t \overline{\frac{\eta\,|v^0|^2 \, + \, \eta\alpha}{2}} \; = -\mathrm{div}_x \left(\overline{\eta\, v^0}\, q \right).$$

Together with identity (10), this leads to

$$\partial_t \overline{\eta\alpha} \; + \; 2\overline{\eta\beta : \nabla_y v^0} = 0. \tag{11}$$

We will deduce from (11) a Gronwall type inequality on $\gamma = \overline{\eta\,\alpha}$. Note that

$$\begin{aligned} \nabla_y \left(\eta\alpha\right) \; &= \; -\nabla_y \left(\eta v^0 \otimes v^0\right) \; + \; \lim_{\varepsilon\to 0} \, \varepsilon \nabla_x \left(\eta_\varepsilon v^\varepsilon \otimes v^\varepsilon\right) \\ &= \; -\nabla_y \left(\eta v^0 \otimes v^0\right) \; + \; \left(\nabla_y \eta\right) \alpha \; + \; 2\,\eta \, \lim_{\varepsilon\to 0} \left(\nabla_x(\varepsilon v^\varepsilon)\right) v^\varepsilon, \end{aligned} \tag{12}$$

where the limit is in the sense of two-scale convergence. As

$$v^0 \in L^\infty\left(\mathbb{R}^+;\, L^q(\Omega \times \mathbb{T}^2)\right), \quad \mathrm{curl}_y v^0 \in L^\infty\left(\mathbb{R}^+ \times \Omega \times \mathbb{T}^2\right),$$

one has

$$\nabla_y \left(\eta v^0 \otimes v^0\right) \in L^\infty\left(\mathbb{R}_+; L^{q/2}\left(\Omega; W^{1,p}(\mathbb{T}^2)\right)\right), \; \forall p < +\infty.$$

The second term in the right hand side belongs to $L^\infty\left(\mathbb{R}_+; L^{q/2}(\Omega \times \mathbb{T}^2)\right)$. Moreover,

$$\left(\nabla_x(\varepsilon v^\varepsilon)\right) v^\varepsilon \text{ is bounded in } L^\infty\left((\mathbb{R}_+; L^p(\Omega)\right)$$

for all $p < q$. Hence, the right hand side belongs to $L^\infty\left(\mathbb{R}_+; L^{q/2}(\Omega \times \mathbb{T}^2)\right)$. Thus, by standard Sobolev imbedding

$$\eta\alpha \in L^\infty\left(\mathbb{R}_+; L^{q/2}\left(\Omega; W^{1,q/2}(\mathbb{T}^2)\right)\right) \;\hookrightarrow\; L^\infty\left(\mathbb{R}_+; L^{q/2}\left(\Omega; L^q(\mathbb{T}^2)\right)\right).$$

Using again (12), we obtain for any $2 < p < q$,

$$\eta\alpha \in L^\infty\left(\mathbb{R}_+; L^{q/2}\left(\Omega; W^{1,p}(\mathbb{T}^2)\right)\right) \;\hookrightarrow\; L^\infty\left(\mathbb{R}_+; L^{q/2}\left(\Omega; C^0(\mathbb{T}^2)\right)\right).$$

Consequently, for almost every x,

$$\|\eta(x,\cdot)\alpha(x,\cdot)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{T}^2)} \;\le\; M(x) < \infty. \tag{13}$$

Let us then point out that $v^0 \in L^2\left(\Omega; C^0(\mathbb{T}^2)\right)$. Hence, the function $x \mapsto v(t,x,x/\varepsilon)$ is measurable (see [13]), and it follows that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \eta_\varepsilon |v^\varepsilon - v^0(t,x,x/\varepsilon)|^2 \;&=\; \eta\,\alpha, \\
\lim_{\varepsilon\to 0} \eta_\varepsilon (v^\varepsilon - v^0(t,x,x/\varepsilon)) \otimes (v^\varepsilon - v^0(t,x,x/\varepsilon)) \;&=\; \eta\,\beta,
\end{aligned}$$

where the limit is in the sense of two-scale convergence. We deduce that

$$\eta|\beta| \;\le\; C\,\eta\,\alpha. \tag{14}$$

We also stress that $\mathrm{curl}_y\, v^0$ being bounded, we have for almost every x,

$$\|\nabla_y v^0(x,\cdot)\|_{L^\infty_t(L^p_y)} \;\le\; N(x)\,p, \quad N(x) < \infty,$$

for all finite p. Equation (11) gives, thanks to (13),(14), for almost every x, for all finite p,

$$\begin{aligned}
\partial_t \gamma(t,x) \;&\le\; C \int_{\mathbb{T}^2} (\eta\alpha)(t,x,y)|\nabla_y v^0|(t,x,y)\,dy \\
&\le\; C\,M(x)^{1/p}\,\gamma(t,x)^{1-1/p}\,\|\nabla_y v^0(x,\cdot)\|_{L^\infty_t(L^p_y)} \;\le\; C(x)\,p\,\gamma(t,x)^{1-1/p}.
\end{aligned}$$

and we conclude as in [10, p321]. As $\gamma|_{t=0}(x) = 0$ for almost every x, we obtain $\gamma = 0$, and thus $\alpha = 0$, $\beta = 0$, which ends the proof.

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