J. math. fluid mech. XX (2004) 1–22
1422-6928/04/010001-22

DOI 10.1007/s00021-004-0115-9

Journal of Mathematical Fluid Mechanics

Formal Derivation of Boundary Layers in Fluid Mechanics

David Gerard-Varet

Communicated by G. Iooss

Abstract. Boundary layers appear in various areas of fluid dynamics, as oceanology, meteorology, or magnetohydrodynamics (MHD). Some of them are already mathematically well known, like the Ekman layers. Many others remain unstudied, and can be much more complex. The aim of this paper is to give both a unified presentation of the main boundary layers, and a simple method to derive their size and equations. This method, based on elementary formal computations, is then applied to classical geophysical systems. We recover in this way many results contained in the physical literature.

Mathematics Subject Classification (2000). ?

Keywords. ?

1. Introduction

One of the main features in oceanography and meteorology, and also in magneto-hydro-dynamics is the presence of one or more small parameters. Typically in oceanography, after appropriate time and space rescaling, the rotation speed of the Earth (which creates the Coriolis force) is pretty large (10^2 to 10^4), the aspect ratio (ratio between the depth and the length) is small (like a few kilometers over several thousands kilometers), parameters describing the stratification (like the so called Brunt Vassaila frequency) are large. In MHD, in the study of the Earth magnetic field, the rotation speed is even larger (after rescaling, something like 10^8), the strength of the magnetic field is very important (10^8 also).

In the interior of the domain these small parameters lead to some reduced behavior. For instance, in highly rotating fluids, the velocity field is invariant in the direction of the rotation axis: this result is known as the Taylor–Proudman theorem (see textbook [12]). This reduced behavior is often incompatible with boundary conditions. This leads to boundary layers, small zones near the boundary where the fluid velocity changes rapidly, in order to satisfy the boundary conditions (typically the no-slip condition at a rigid surface).

In some cases, the derivation of the layers is easy, as in the study of viscous perturbations of hyperbolic systems (see [11, 15]). But in most problems, the situation is more complex, involving a variety of length scales and equations. A typical example is the evolution of an incompressible viscous fluid in a highly rotating domain. Governing equations are Navier–Stokes equations, with a large Coriolis term

$$\partial_t u + u \cdot \nabla u + \frac{\mathbf{e} \times u}{\varepsilon} + \frac{\nabla p}{\varepsilon} - \frac{E}{\varepsilon}\Delta u = 0, \tag{1.1}$$

$$\nabla \cdot u = 0. \tag{1.2}$$

where $\mathbf{e} = (0, 0, 1)^t$ is a given fixed vector, and E and ε are two small parameters, called Ekman and Rossby numbers. A traditional scaling is E proportional to ε^2, relevant to the Earth's liquid core (see [7] for details). For this system, only the case of a flat horizontal boundary has been fully studied, with the development of the Ekman layer with size $E^{1/2}$. We refer to monograph [20] or articles [14, 19, 3] for rigorous justification. In more complex geometries, following the fundamental papers of Stewartson [24, 25], many other layers appear: for instance, between two concentric spheres, boundary layers of size $E^{1/2}$, $E^{1/3}$, $E^{1/4}$, $E^{2/5}, \dots$ are expected near the inner sphere and the cylinder circumscribing it (see figure 1).

Classically, the derivation of such boundary layers is done through huge computations, with many technicalities. Broadly speaking, two approaches emerge from the literature:

- The first one is analytical: one computes the exact solution, and then performs an asymptotic analysis on it. However, such technique is restrictive (the exact solution is rarely computable), and often tedious (cf. [24] with Bessel functions).
- The second one relies on the so-called "matching asymptotic expansions method": for a general presentation, we refer the reader to [27, 8]. See also [16, 23, 28]. This method has been applied with success to various physical situations, with for instance the treatment of the famous Prandtl layer. It allows a precise study of many singular perturbation problems, in a very general framework (including nonlinear partial differential equations). However, it often leads to heavy computations (see for instance [26] on the rotating fluids system). It is often supplied with refined physical arguments, so as to reduce the number of unknowns.

The aim of this paper is to give a simpler approach of boundary layer problems. We propose a formal method, based on elementary algebraic computations, to derive their size and in some cases their equations. Applied to classical systems of geophysics, this method allows to recover in a simple way many results exposed in the literature. Up to the best of our knowledge, this simple presentation of boundary layer equations has never been detailed in such a framework. We hope this work will help mathematicians to get interested in the numerous existence, unicity and stability related problems. This paper was announced in the proceedings note [10], written with Emmanuel Grenier, in which the ideas were given without any detail.

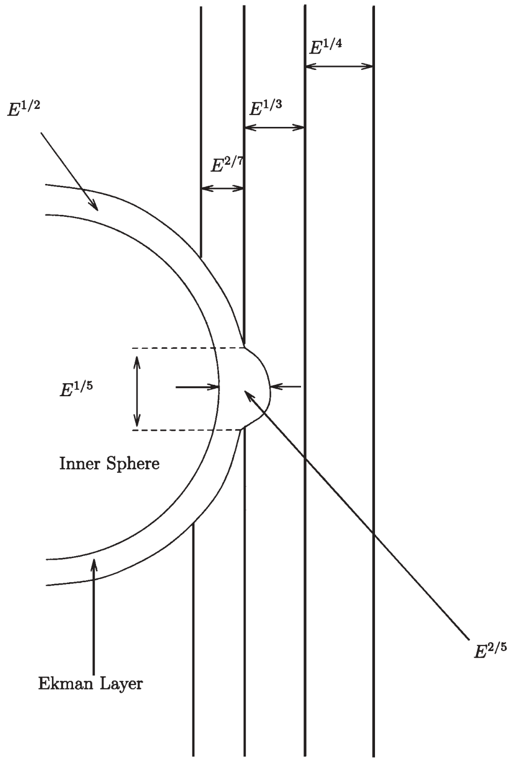

FIG. 1. Boundary layers of rotating fluids, near a sphere and at the circumscribing cylinder (following Stewartson [25]).

The rest of this paper is structured as follows: section 2 is devoted to the general presentation of the method. Sections 3 to 6 detail some of its application to flat boundary layers. Finally, section 7 extends this formal derivation to spherical layers. Note that most of the formal results in the flat case can be rigorously justified (see on different geophysical systems articles [4, 14, 5, 6, 9]), however the spherical case remains widely open.

2. Presentation

This part presents the method in the general case, emphasizing the main ideas underlying it. In what follows, we start with systems of the form

$$\mathcal{A}^\varepsilon U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon) = F^\varepsilon, \tag{2.1}$$

where F^ε is some smooth forcing term, $\mathcal{A}^\varepsilon$ is a matricial linear differential operator and $\mathcal{Q}^\varepsilon$ is the nonlinear part of the equation. Note that equations (1.1), (1.2) can be written this way, with for instance

$$U^\varepsilon = (\, u^\varepsilon p^\varepsilon \,)\,, \quad \mathcal{Q}^\varepsilon(U^\varepsilon) = (\, u^\varepsilon \cdot \nabla u^\varepsilon 0 \,)\,.$$

Let us suppose that equation (2.1) holds in a domain

$$t \in \mathbb{R}_+^*,\ x \in \Omega, \qquad \Omega = \Big\{ x = (x_\parallel, x_\perp),\ x_\perp > 0 \Big\}$$

(one flat boundary). As usual in boundary layer problems, we expect solutions U^ε of (2.3) to have an asymptotic expansion

$$U^\varepsilon(t,x) = \sum_{j=0}^{+\infty} \left(\varepsilon^\beta\right)^j U^j \left(t, x, \frac{x_\perp}{\varepsilon^{\alpha_1}}, \dots, \frac{x_\perp}{\varepsilon^{\alpha_n}}\right) \tag{2.2}$$

where ε^{α_1},..., ε^{α_n} are the possible sizes of the layer at $\partial\Omega$, ε^β is the characteristic size of the U^ε - components, and where the profiles U^j have the decomposition

$$U^j(t, x, \theta_1, \dots, \theta_n) = U^j_{int}(t,x) + \sum_{k=1}^{n} U^j_k \left(t, x_\parallel, \theta_k\right)$$

with an interior term U^j_{int} describing the solution far from the boundary, and a boundary layer part $\sum U^j_k$. The main questions relative to boundary layers are then:

- What are the possible $\alpha_1, \dots, \alpha_n, \beta$?
- What are the equations satisfied by the U^j_k, or at least by U^0_k ?

The first idea is that boundary layers already appear on linear equations, therefore in a first step we can dismiss the term $\mathcal{Q}^\varepsilon$. This is a very classical approach, underlying most of the physical studies. This is often[1] *a posteriori* justified, since in many cases the nonlinear term appears to be a higher order perturbation of the linear case. However this nonlinear term is important when we look at stability issues. It is this term which destabilizes many flows. In a crude way we can say that the nonlinear term does not create the boundary layer, but may destabilize it. Therefore, we look at equation

$$\mathcal{A}^\varepsilon U^\varepsilon = F^\varepsilon. \tag{2.3}$$

In most cases, the external force acts in the interior of the domain, and is not located in the boundary layers. In those cases, the boundary layer part of (2.2) formally satisfies the homogeneous equation associated with (2.3) (of course, if F^ε is supported in small areas near the boundaries, F^ε can create itself boundary layers, whose sizes do not appear in the force free case. Then, the sizes of these

[1] Some boundary layers are genuinely nonlinear, and do not allow linearization (like the famous Prandtl layer [17, 13]). But to our knowledge, such layers are quite singular: generically, boundary layers are already "within the linear systems".

layers are simply given by the sizes of the support of F^ε). Thus, in order to derive the boundary layers, it is natural to consider equation

$$\mathcal{A}^\varepsilon U^\varepsilon = 0. \tag{2.4}$$

The main point is the following: for the systems we know, the study of (2.4) is sufficient to get the main features of the layers. To go back to (2.1) then needs a careful analysis and justification, but on clearly identified layers and with clearly given sizes of the various components of U^ε, which greatly helps. To study (2.4), we make a Laplace-Fourier analysis, considering solutions of (2.4) of the form

$$U^\varepsilon = e^{i\tau^\varepsilon t + i\xi^\varepsilon x} V^\varepsilon. \tag{2.5}$$

The use of such modal solutions is not new, as it appears in stability issues or in the study of hyperbolic systems (see for instance [21] for application to linear geometric optics). In our context, it will lead us to the desired sizes and equations.

Size of the layers

Let A^ε the symbol of $\mathcal{A}^\varepsilon$ and $a^\varepsilon = \det A^\varepsilon$. We introduce the characteristic manifold of $\mathcal{A}^\varepsilon$,

$$\sigma^\varepsilon = \Big\{ (\tau^\varepsilon, \xi^\varepsilon) \quad | \quad a^\varepsilon(\tau^\varepsilon, \xi^\varepsilon) = 0 \Big\}.$$

All the nontrivial modal solutions of (2.4) satisfy $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$.

Broadly speaking, boundary layers will correspond to $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$ with

$$\mathcal{I}m\,(\xi^\varepsilon_\perp) \xrightarrow[\varepsilon \to 0]{} +\infty.$$

Their sizes will then be given by $|\mathcal{I}m\,(\xi^\varepsilon_\perp)|^{-1}$. To be more precise, we introduce the following definition

Definition 2.1. Let $\alpha > 0$. We shall say that ε^α is a boundary layer size if there exists $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$, with

- $(\tau^\varepsilon, \xi^\varepsilon_\parallel)$ independent of ε
- $\xi^\varepsilon_\perp \sim \frac{C}{\varepsilon^\alpha}$, with $\mathcal{I}m\,(C) > 0$, as ε goes to zero.

Remark. The last condition on the asymptotic behavior of $\xi^\varepsilon_\perp$ is not restrictive. Indeed, if $(\tau^\varepsilon, \xi^\varepsilon_\parallel)$ is independent of ε, $p^\varepsilon = a^\varepsilon(\tau^\varepsilon, \xi^\varepsilon_\parallel, \cdot)$ will be in all interesting cases a polynomial, with coefficients meromorphic in ε. Classical complex analysis (cf. [1]) shows then that for small enough $\varepsilon > 0$, the roots of this polynomial are representable by smooth functions $\xi^1_\perp(\varepsilon), \dots, \xi^s_\perp(\varepsilon)$ with

$$\xi^i_\perp(\varepsilon) \sim \frac{C_i}{\varepsilon^{\alpha_i}}, \quad \alpha_i \in \mathbb{Q}, \quad C_i \in \mathbb{C}, \quad \varepsilon \to 0.$$

Thus, the derivation of the sizes of the layers reduces to the search of asymptotic behaviors of solutions of $a^\varepsilon = 0$.

Remark. Instead of looking at characteristic length scales (boundary layers), we can consider in the same way characteristic time scales (time layers and oscillations), by swapping the roles of τ^ε and $\xi_\perp^\varepsilon$. Thus, we may study $(\tau^\varepsilon, \xi_\perp^\varepsilon) \in \sigma^\varepsilon$ with

- ξ^ε independent of ε, $\mathcal{I}m\,(\tau^\varepsilon) \to \infty$: initial (time) boundary layer
- ξ^ε independent of ε, $\mathcal{R}e\,(\tau^\varepsilon) \to \infty$: high frequency oscillations in the domain, with frequency of order $\mathcal{R}e\,(\tau^\varepsilon)$.

We may also look for mixed behaviors, for instance $\mathcal{I}m\,(\tau^\varepsilon) \to \infty$ and $\mathcal{R}e\,(\tau^\varepsilon) \to \infty$, which corresponds to an oscillatory boundary layer.

Equations of the layers

Once we have the size of the layers, the next step is to find their equations. Let ε^α a boundary layer size. Let $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$ satisfying the two conditions of definition 1. Let $\pi(\tau^\varepsilon, \xi^\varepsilon)$ the spectral projection of $A^\varepsilon(\tau^\varepsilon, \xi^\varepsilon)$ on its kernel. It is clear that U^ε is a modal solution of (2.4) iff

$$\pi(\tau^\varepsilon, \xi^\varepsilon) V^\varepsilon = V^\varepsilon.$$

Suppose that the range of $\pi(\tau^\varepsilon, \xi^\varepsilon)$ is one-dimensional (for small enough $\varepsilon > 0$). Then it is easy to show that up to a multiplication by an arbitrary function of ε, U^ε is uniquely defined and has the asymptotic behaviour

$$U^\varepsilon(t,x) \sim \begin{pmatrix} \varepsilon^{\beta_1} & & \\ & \ddots & \\ & & \varepsilon^{\beta_n} \end{pmatrix} \tilde{U}^0\left(t, x_\parallel, \frac{x_\perp}{\varepsilon^\alpha}\right), \quad \varepsilon \to 0$$

where $\beta_1, \dots, \beta_n \in \mathbb{Q}$ so that we get the size ε^{β_i} of each component of this boundary layer solution.

To get the equations of the layer, we rescale the symbol by:

$$\tilde{A}^\varepsilon(\tau, \xi) = A^\varepsilon\left(\tau, \xi_\parallel, \frac{\xi_\perp}{\varepsilon^\alpha}\right) \begin{pmatrix} \varepsilon^{\beta_1} & & \\ & \ddots & \\ & & \varepsilon^{\beta_n} \end{pmatrix}.$$

In all our systems, $\tilde{A}^\varepsilon$ will have a finite Laurent development in a certain power of ε. The leading coefficient of this development will be a symbol in (τ, ξ), which will give the equations of the layer. We refer to next sections for extensive application. At this point, the approach is completely elementary. Its interest lies in the fact that it provides a very quick and easy way to derive a large number of classical and not so classical boundary layers.

In the following of the paper, we apply our method to classical geophysical systems, and recover in a simple way many results exposed in the literature.

3. Rotating fluids

We consider in this section equations (1.1), (1.2) of a rotating fluid, with the physical scaling $E = \varepsilon^2$. After we drop the $u \cdot \nabla u$ term, this system can be written

$$\mathcal{A}^\varepsilon U^\varepsilon = 0, \quad U^\varepsilon = \begin{pmatrix} u^\varepsilon \\ p^\varepsilon \end{pmatrix}. \tag{3.1}$$

where $\mathcal{A}^\varepsilon$ is the differential operator with matricial symbol ($\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$):

$$A^\varepsilon(\tau, \xi) = \begin{pmatrix} i\tau + \varepsilon\xi^2 & -\varepsilon^{-1} & 0 & \varepsilon^{-1}\xi_1 \\ \varepsilon^{-1} & i\tau + \varepsilon\xi^2 & 0 & \varepsilon^{-1}\xi_2 \\ 0 & 0 & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}\xi_3 & 0 \end{pmatrix}$$

The determinant, obtained by expansion along the last row, is

$$a^\varepsilon(\tau, \xi) = (i\tau + \varepsilon\xi^2)^2\xi^2 + \frac{\xi_3^2}{\varepsilon^2}. \tag{3.2}$$

Remark. Equation $a^\varepsilon = 0$ is the "algebraic translation" of the fundamental equation on pressure:

$$(\partial_t - \varepsilon\Delta)^2 \Delta p + \varepsilon^{-2}\frac{\partial^2 p}{\partial z^2} = 0$$

derived by Greenspan in [12].

As explained in section 2, we derive the different boundary layers by looking at elements $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma_\varepsilon$ satisfying different asymptotics as ε goes to zero. Note that every solution $V = \begin{pmatrix} u \\ p \end{pmatrix}$ of $A^\varepsilon(\tau, \xi)V = 0$ satisfies relations

$$u_1 = \frac{-\varepsilon^{-1}\xi_1\gamma + \varepsilon^{-2}\xi_2}{\gamma^2 + \varepsilon^{-2}} p, \quad u_2 = \frac{-\varepsilon^{-1}\xi_2\gamma + \varepsilon^{-2}\xi_1}{\gamma^2 + \varepsilon^{-2}} p, \quad \gamma u_3 = -\varepsilon^{-1}\xi_3 p. \tag{3.3}$$

3.1. Ekman layers

We first derive the possible horizontal boundary layers. They correspond to elements $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$ with $(\tau^\varepsilon, \xi_1^\varepsilon, \xi_2^\varepsilon) = (\tau, \xi_1, \xi_2)$ independent of ε, and

$$\mathcal{I}m\,(\xi_3^\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty.$$

We deduce from equation (3.2) as ε goes to zero

$$\varepsilon^2(\xi_3^\varepsilon)^6 \sim -\frac{(\xi_3^\varepsilon)^2}{\varepsilon^2} \tag{3.4}$$

$$\iff \quad \xi_3^\varepsilon \sim \frac{\pm 1 \pm i}{\sqrt{2}\varepsilon} \tag{3.5}$$

which gives one boundary layer size $\varepsilon = E^{1/2}$. This is the so-called Ekman layer.

Let $(\tau^\varepsilon, \xi^\varepsilon)$ as above, ξ_3^ε satisfying (3.5). Let V^ε in $\ker(A^\varepsilon(\tau^\varepsilon, \xi^\varepsilon))$. Using (3.3), V^ε satisfies (modulo a multiplication by an arbitrary function of ε):

$$u_1 \sim C_1, \quad u_2 \sim C_2, \quad u_3 \sim C_3\, \varepsilon, \quad p \sim C_4\, \varepsilon^2, \quad \varepsilon \to 0.$$

where the C_i's denote various constants. Back to the corresponding modal solution U^ε, we have:

$$U^\varepsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & \\ & & & \varepsilon^2 \end{pmatrix} \left(\tilde{U}^0(\cdot, \varepsilon^{-1} z) + \varepsilon \tilde{U}^1(\cdot, \varepsilon^{-1} z) + \dots \right).$$

Equation on $U^0 = \begin{pmatrix} u^0 \\ p^0 \end{pmatrix}$ is obtained by rescaling the matricial symbol: if we set

$$\tilde{A}^\varepsilon(\tau, \xi_1, \xi_2, \xi_3) = \tilde{A}^\varepsilon(\tau, \xi_1, \xi_2, \varepsilon^{-1}\xi_3) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & \\ & & & \varepsilon^2 \end{pmatrix}, \tag{3.6}$$

we have $\tilde{A}^\varepsilon = \tilde{A}^0 + \varepsilon \tilde{A}^1 + \dots$, with leading symbol:

$$\tilde{A}^0 = \begin{pmatrix} \xi_3^2 & -1 & 0 & 0 \\ 1 & \xi_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix},$$

which corresponds to equations

$$\tilde{u}_2^0 + \partial_Z^2 \tilde{u}_1^0 = 0, \tag{3.7}$$

$$\tilde{u}_1^0 - \partial_Z^2 \tilde{u}_2^0 = 0, \tag{3.8}$$

$$\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 + \partial_Z \tilde{u}_3^0 = 0. \tag{3.9}$$

All these results match the derivation obtained in [20] through an asymptotic expansion. Already in this simple case, the computations are much faster than the usual Ansatz study.

Remark. We may compute the size of the components of the nonlinear term. We find

$$u \cdot \nabla u_1 = O(1), \quad u \cdot \nabla u_2 = O(1), \quad u \cdot \nabla u_3 = O(\varepsilon).$$

It is small compared to the other terms involved in the equation of the layer (since $-\varepsilon \Delta u = O(\varepsilon^{-1})$), which justifies a posteriori that we neglect the nonlinear terms in the derivation of the equation of the boundary layer.

3.2. Vertical layers

We look for elements $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$ with τ, ξ_2, ξ_3 independent of ε, and

$$\mathcal{I}m\,(\xi_1^\varepsilon) \xrightarrow[\varepsilon\to 0]{} \infty.$$

Looking at equation $a^\varepsilon = 0$, there are two cases to consider

– $\xi_3 \neq 0$.

In this case, we get

$$\varepsilon^2 \xi_1^6 \sim -\frac{\xi_3^2}{\varepsilon^2},$$

which shows that $\varepsilon^{2/3} = E^{1/3}$ is a boundary layer size. We can proceed as in the case of the Ekman layer. We get

$$u_1 \sim C_1\,\varepsilon^{2/3}, \quad u_2 \sim C_2, \quad u_3 \sim C_3, \quad p \sim C_4\,\varepsilon^{2/3}, \quad \varepsilon \to 0,$$

so that the corresponding modal solution has the expansion

$$U^\varepsilon = \begin{pmatrix} \varepsilon^{2/3} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varepsilon^{2/3} \end{pmatrix} \left(\tilde{U}^0(\cdot, \varepsilon^{-2/3} z) + \varepsilon^{2/3} \tilde{U}^1(\ldots, \varepsilon^{-2/3} z) + \ldots \right).$$

After rescaling of the matricial symbol A^ε as in (3.6) we get

$$\begin{aligned} \tilde{u}_2^0 - \partial_X \tilde{p}^0 &= 0, \\ \tilde{u}_1^0 + \partial_y \tilde{p}^0 - \partial_x^2 \tilde{u}_2^0 &= 0, \\ \partial_z \tilde{p}^0 + \partial_X^2 \tilde{u}_3^0 &= 0, \\ \partial_X \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 + \partial_z \tilde{u}_3^0 &= 0. \end{aligned}$$

It leads to a single equation on pressure

$$\frac{\partial^2 \tilde{p}^0}{\partial Z^2} + \frac{\partial^6 \tilde{p}^0}{\partial X^6} = 0. \tag{3.10}$$

Remark. Computing again the size of the components of the nonlinear term, we find

$$u \cdot \nabla u_1 = O(\varepsilon^{2/3}), \quad u \cdot \nabla u_2 = O(1), \quad u \cdot \nabla u_3 = O(1)$$

which is still small compared to the other terms involved in the equation.

– $\xi_3 = 0$.

In this case, we get $\xi_1^\varepsilon \sim \pm\sqrt{\dfrac{\tau}{\varepsilon}}$. This layer has size $\varepsilon^{1/2} = E^{1/4}$. We thus recover the two scales announced in [24]. For this last layer, we get

$$u_1 \sim C_1\,\varepsilon^{1/2}, \quad u_2 \sim C_2, \quad p \sim C_3\,\varepsilon^{1/2}.$$

Note that the size of u_3 is free, which is natural since we look at modal solutions independent of z ($\xi_3 = 0$). Leading equations of the layer are then

$$\tilde{u}_2^0 = \partial_X \tilde{p} \tag{3.11}$$

$$\tilde{u}_1^0 = -\partial_y \tilde{p} \tag{3.12}$$

$$\partial_X \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 = 0. \tag{3.13}$$

3.3. Time scales

As mentioned in section 2, we can through our method investigate time scales of the problem instead of length scales. We consider $\xi^\varepsilon = \xi$ real (Fourier mode) and independent of ε. We look for solutions of $a^\varepsilon = 0$ with τ^ε going to infinity: we get the asymptotic behaviour

$$\tau \sim \pm \frac{i}{\varepsilon} \frac{|\xi_3|}{|\xi|}$$

which corresponds to high frequency oscillations of frequency ε^{-1}. These are the Rossby waves, described from a physical viewpoint in [20], and from a mathematical viewpoint in [2, 19, 3].

4. Hartmann layers

4.1. Governing equations

We consider an incompressible fluid under the action of a vertical magnetic field of high intensity. Neglecting the current displacements and under the assumption of a small magnetic Reynolds number, the dimensionless equations are:

$$\partial_t u + u \cdot \nabla u + \frac{\nabla p}{\varepsilon} - \Delta u + \frac{\mathbf{e} \times j}{\varepsilon} = 0, \tag{4.1}$$

$$j = \nabla \varphi + u \times \mathbf{e}, \tag{4.2}$$

$$\nabla \cdot u = 0, \tag{4.3}$$

$$\nabla \cdot j = 0 \tag{4.4}$$

(see [18] for details). $\mathbf{e} = (0, 0, 1)$, u is the velocity field, p the pressure, j the current density, φ the electromagnetic potential. We have set $\varepsilon = M^{-2}$, where M is the Hartmann number, proportional to the average magnetic field, so that $\varepsilon \ll 1$. As previously, in a first step we neglect the nonlinear term, and consider the simplified linear system:

$$\begin{aligned} \partial_t u + \frac{\nabla p}{\varepsilon} - \Delta u + \frac{\mathbf{e}\times j}{\varepsilon} &= 0, \\ j &= \nabla\varphi + u\times\mathbf{e}, \\ \nabla\cdot u &= 0, \\ \nabla\cdot j &= 0. \end{aligned}$$

This system can be written

$$\mathcal{A}^\varepsilon U^\varepsilon = 0, \quad U^\varepsilon = (\, u^\varepsilon j^\varepsilon p^\varepsilon \varphi^\varepsilon \,)\,.$$

The symbol A^ε is

$$\begin{pmatrix} i\tau+\xi^2 & 0 & 0 & 0 & -\varepsilon^{-1} & 0 & i\xi_1 & 0 \\ 0 & i\tau+\xi^2 & 0 & \varepsilon^{-1} & 0 & 0 & i\xi_2 & 0 \\ 0 & 0 & i\tau+\xi^2 & 0 & 0 & 0 & i\xi_3 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -i\xi_1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & -i\xi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -i\xi_3 \\ \varepsilon^{-1}i\xi_1 & \varepsilon^{-1}i\xi_2 & \varepsilon^{-1}i\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1}i\xi_1 & \varepsilon^{-1}i\xi_2 & \varepsilon^{-1}i\xi_3 & 0 & 0 \end{pmatrix}.$$

We will look for solutions

$$V = \begin{pmatrix} u \\ j \\ p \\ \varphi \end{pmatrix} \neq 0, \qquad \text{of} \qquad A^\varepsilon V = 0. \tag{4.5}$$

Direct computation of the determinant is possible, however it is easier to express j as a function of u, which reduces the problem to a 4 by 4 matrix. More precisely, incompressibility condition on j is written

$$\xi_1 j_1 + \xi_2 j_2 + \xi_3 j_3 = 0.$$

Using expressions of j given by lines 4, 5, 6 of system $A^\varepsilon V = 0$, we get:

$$-\xi^2\varphi = -i\xi_1 u_2 + i\xi_2 u_1.$$

For the boundary layer solutions, $\xi^2 \neq 0$, so that we can divide last equation by ξ^2. We inject this expression of φ in lines 4,5,6 of system $A^\varepsilon V = 0$ and find

$$j_1 = \frac{\xi_2^2+\xi_3^2}{\xi^2}u_2 + \frac{\xi_1\xi_2}{\xi^2}u_1, \tag{4.6}$$

$$j_2 = -\frac{\xi_1^2+\xi_3^2}{\xi^2}u_1 - \frac{\xi_1\xi_2}{\xi^2}u_2, \tag{4.7}$$

$$j_3 = \frac{\xi_3}{\xi^2}(\xi_2 u_1 - \xi_1 u_2). \tag{4.8}$$

Then, we plug these expressions of j_1, j_2, j_3 in lines 1, 2, 3 of system $A^\varepsilon V = 0$. Using the incompressibility condition on u $\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 = 0$, we finally get a reduced system on u and p:

$$\begin{pmatrix} \alpha & 0 & i\beta\xi_1 & \varepsilon^{-1}i\xi_1 \\ 0 & \alpha & i\beta\xi_2 & \varepsilon^{-1}i\xi_2 \\ 0 & 0 & \alpha + i\beta\xi_3 & \varepsilon^{-1}i\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}i\xi_3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{pmatrix} = 0,$$

where

$$\alpha = i\tau + \xi^2 + \frac{\xi_3^2}{\epsilon\xi^2}, \quad \beta = i\frac{\xi_3}{\varepsilon\xi^2}.$$

If we set:

$$p' := p + \beta u_3,$$

this system reduces to

$$\begin{pmatrix} \alpha & 0 & 0 & \varepsilon^{-1}\xi_1 \\ 0 & \alpha & 0 & \varepsilon^{-1}\xi_2 \\ 0 & 0 & \alpha & \varepsilon^{-1}\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}\xi_3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p' \end{pmatrix} = 0. \tag{4.9}$$

Note that this system has non trivial solutions $\binom{u}{p}$ iff system (4.5) has non zero solutions V. Thus, computing the determinant of the last system, we have that equation $a^\varepsilon = 0$ is equivalent to equation:

$$i\tau + \xi^2 + \frac{\xi_3^2}{\varepsilon\xi^2} = 0. \tag{4.10}$$

4.2. Horizontal and vertical layers

Let (τ, ξ_1, ξ_2) fixed. Solutions ξ_3^ε with imaginary part going to infinity (i.e. horizontal layers) satisfy

$$\xi_3 \sim \pm i\sqrt{\varepsilon}, \quad \varepsilon \to 0,$$

i.e. the boundary layer size is $\varepsilon^{1/2}$. It is the Hartmann layer, physically described in [22].

In the case $\mathcal{I}m\, \xi_1^\varepsilon \to \infty$ (vertical layers), we have

$$(\xi_1^\varepsilon)^4 \sim -\varepsilon^{-1}\xi_3^2,$$

which leads to boundary layer size $\varepsilon^{1/4}$. Note that in this case, the kernel of $A^\varepsilon(\tau, \xi^\varepsilon)$ has dimension ≥ 2, so that there are different possible sizes (and equations) for our modal solutions.

4.3. Time layers

As in the case of rotating fluids, we look at $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma^\varepsilon$ with ξ^ε independent of ε and τ^ε going to infinity. We obtain from (4.10): $i\tau \sim \dfrac{\xi_3^2}{\xi^2\varepsilon}$. Contrary to the previous system, there is no oscillation, but an initial time layer of size ε. For more physical insight, we refer to [22, 18].

5. MHD

5.1. Governing equations

We consider in this section an incompressible viscous fluid, at both high rotation and strong magnetic field, mixing the main features of the two previous sections. It is of great geophysical interest, as it is a first step in the understanding of the magnetohydrodynamic flow in the Earth's core. At very small magnetic Reynolds number, and under the "geological" scalings (see [5]), governing equations in a dimensionless form are:

$$\partial_t u + u \cdot \nabla u + \frac{\nabla p}{\varepsilon} - \Delta u + + \frac{\mathbf{e} \times u}{\varepsilon} + \frac{\Lambda \mathbf{e} \times j}{\varepsilon} = 0, \tag{5.1}$$

$$j = \nabla \varphi + u \times \mathbf{e}, \tag{5.2}$$

$$\nabla \cdot u = 0, \tag{5.3}$$

$$\nabla \cdot j = 0, \tag{5.4}$$

where $\mathbf{e} = (0, 0, 1)$ and Λ is a constant. We will not take into account the nonlinearity and consider system

$$\begin{aligned} \partial_t u + \frac{\nabla p}{\varepsilon} - \Delta u + + \frac{\mathbf{e} \times u}{\varepsilon} + \frac{\Lambda \mathbf{e} \times j}{\varepsilon} &= 0, \\ j &= \nabla \varphi + u \times \mathbf{e}, \\ \nabla \cdot u &= 0, \\ \nabla \cdot j &= 0. \end{aligned}$$

5.2. Symbol

Last system can be written

$$\mathcal{A}^\varepsilon U = 0, \quad U = \begin{pmatrix} u \\ j \\ p \\ \varphi \end{pmatrix}.$$

The symbol A^ε is

$$\begin{pmatrix} i\tau+\xi^2 & -\varepsilon^{-1} & 0 & 0 & -\varepsilon^{-1} & 0 & i\varepsilon^{-1}\xi_1 & 0 \\ \varepsilon^{-1} & i\tau+\xi^2 & 0 & \varepsilon^{-1} & 0 & 0 & i\varepsilon^{-1}\xi_2 & 0 \\ 0 & 0 & i\tau+\xi^2 & 0 & 0 & 0 & i\varepsilon^{-1}\xi_3 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -i\xi_1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & -i\xi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -i\xi_3 \\ \varepsilon^{-1}i\xi_1 & \varepsilon^{-1}i\xi_2 & \varepsilon^{-1}i\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1}i\xi_1 & \varepsilon^{-1}i\xi_2 & \varepsilon^{-1}i\xi_3 & 0 & 0 \end{pmatrix}.$$

Computations are then similar to those of the previous section: we come to a reduced system:

$$\begin{pmatrix} \alpha & -\epsilon^{-1} & 0 & \varepsilon^{-1}\xi_1 \\ \epsilon^{-1} & \alpha & 0 & \varepsilon^{-1}\xi_2 \\ 0 & 0 & \alpha & \varepsilon^{-1}\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{pmatrix} = 0$$

where

$$\alpha = i\tau + \epsilon\xi^2 + \frac{\Lambda\xi_3^2}{\epsilon\xi^2}.$$

Equation $a^\varepsilon = 0$ is then equivalent to equation:

$$\left(i\tau + \epsilon\xi^2 + \frac{\Lambda\xi_3^2}{\epsilon\xi^2}\right)^2 \xi^2 + \frac{\xi_3^2}{\epsilon^2} = 0. \tag{5.5}$$

We also deduce from the reduced system that non-trivial elements of $\ker(A^\varepsilon)$ satisfy

$$u_1 = \frac{-\varepsilon^{-1}\xi_1\alpha + \varepsilon^{-2}\xi_2}{\alpha^2+\epsilon^{-2}}p, \quad u_2 = \frac{-\varepsilon^{-1}\xi_2\alpha + \varepsilon^{-2}\xi_1}{\alpha^2+\epsilon^{-2}}p, \quad \alpha u_3 = -\varepsilon^{-1}\xi_3 p. \tag{5.6}$$

5.3. Horizontal layers

We do not detail computations, as they are similar to those of the previous sections. We get the unique boundary layer size ε. Up to a multiplicative function of ε, we have

$$u_1 \sim C_1, \quad u_2 \sim C_2, \quad u_3 \sim C_3\varepsilon, \quad p \sim C_4\varepsilon^2,$$

$$j_1 \sim C_5, \quad j_2 \sim C_6, \quad j_3 \sim C_7\varepsilon, \quad \varphi \sim C_8\varepsilon^2,$$

and equations are:

$$
\begin{aligned}
\tilde{u}_2^0 - \Lambda \tilde{u}_1^0 + \partial_Z^2 \tilde{u}_1^0 &= 0, \\
\tilde{u}_1^0 - \Lambda \tilde{u}_2^0 + \partial_Z^2 \tilde{u}_2^0 &= 0, \\
\tilde{j}_0^1 - \tilde{u}_0^2 &= 0, \\
\tilde{j}_0^2 + \tilde{u}_0^1 &= 0, \\
\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 + \partial_Z \tilde{u}_3^0 &= 0, \\
\partial_x \tilde{j}_1^0 + \partial_y \tilde{j}_2^0 + \partial_Z \tilde{j}_3^0 &= 0.
\end{aligned}
$$

5.4. Vertical layers

The computations are similar to the pure rotating case. There are two possible sizes of boundary layers, $\varepsilon^{2/3}$ and $\varepsilon^{1/2}$, with the same kind of equations.

6. Munk and Stommel layers

We consider in this section an homogeneous model of wind-driven ocean circulation. Because of the Taylor–Proudman theorem, the evolution of the flow is mostly two-dimensional. As explained in monograph [20] (or rigorously derived in article [6]), a simple but pertinent system to describe the evolution of the horizontal velocity $u(x,y) = (u_1(x,y), u_2(x,y))$ is

$$
\partial_t \omega + u \cdot \nabla \omega + \frac{r}{2}\omega + \beta u_2 - \nu \Delta \omega - \beta \text{curl } \psi = 0,
$$

$$
\omega = \partial_x u_2 - \partial_y u_1, \quad \nabla \cdot u = 0.
$$

$(r/2)\omega$ corresponds to what physicists call “the Ekman pumping”: it is a dissipative term due to the Ekman layer which develops at the bottom of the ocean.The βu_2 term comes from the variation of the Coriolis force with respect to the attitude, and βcurl ψ is a forcing term created by the wind. To remain in the scope of the method, we consider as before the linear part of the equation,

$$
\partial_t \omega + \frac{r}{2}\omega + \beta u_2 - \nu \Delta \omega = \beta \text{curl}(\psi), \tag{6.1}
$$

$$
\nabla \cdot u = 0. \tag{6.2}
$$

System (6.1), (6.2) can be written

$$
\mathcal{A} U = F, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad F = \begin{pmatrix} \text{curl } (\psi) \\ 0 \end{pmatrix},
$$

where $\mathcal{A}$ has symbol

$$
\beta^{-1} \begin{pmatrix} \tau\xi_2 - i\nu\xi^2\xi_2 - ir/2\xi_2 & -\tau\xi_1 + \beta + i\nu\xi^2\xi_1 + ir/2\xi_1 \\ \xi_1 & \xi_2 \end{pmatrix}.
$$

We will look for boundary layers in the x-direction under different scalings.

6.1. Munk layer

We consider the scaling: ν, r and $\beta \gg 1$. If we set $\varepsilon = \beta^{-1}$, the symbol $A = A^\varepsilon$ is linear with coefficients holomorphic in ε. Equation $a^\varepsilon = 0$ is

$$\tau\xi^2 - i\nu\xi^4 - i\frac{r}{2}\xi^2 - \frac{\xi_1}{\varepsilon} = 0. \tag{6.3}$$

If τ^ε and ξ_2^ε are fixed independent of ε, solutions ξ_1^ε whose imaginary part goes to infinity satisfy

$$\nu(\xi_1^\varepsilon)^4 \sim i\varepsilon^{-1}\xi_1,$$

which leads to the boundary layer size $\varepsilon^{1/3}$. This layer is known as the Munk layer (see [20]). Physically, it is responsible for the western intensification of boundary currents (Gulf stream, Kuroshio current, Agulhas current, ... see [20] for details).

If $\xi_2 \neq 0$, solutions V of $A^\varepsilon V$ satisfy

$$u_1 \sim C\,\varepsilon^{1/3}u_2, \quad \varepsilon \to 0.$$

Equations of this layer are then

$$\tilde{u}_2^0 - \nu\partial_X^3\tilde{u}_2^0 = 0, \tag{6.4}$$
$$\partial_X\tilde{u}_1^0 + \partial_y\tilde{u}_2^0 = 0. \tag{6.5}$$

Remarks. 1. One may check easily that the size of the nonlinearity $u \cdot \nabla\omega$ is small compared to the other terms of the equation.

2. If we introduce the stream function p associated to $(\tilde{u}_1^0, \tilde{u}_2^0)$, (6.4), (6.5) are equivalent to equation $-\partial_X^4 p + \partial_X p = 0$. If we add appropriate physical conditions, this allows to determine the boundary layer profile at the western coast (resp. eastern coast) which corresponds in our simplified model to the domain $X > 0$ (resp. $X < 0$). It is then a straightforward computation to see that the normal velocity u_2 at the western coast is $\mathrm{O}(\varepsilon^{-1/3})$ (whereas $u_2 = O(1)$ at the eastern coast). This shows the western intensification of boundary currents. We refer to [6] for complete treatment.

If $\xi_2 = 0$, we get $\tilde{u}_1^0 = 0$, $\partial_y\tilde{u}_2^0 = 0$ and equation

$$\tilde{u}_2^0 - \nu\partial_X^3\tilde{u}_2^0 = 0.$$

It corresponds to a boundary layer term independent of y.

6.2. Stommel layer

This time, we make another choice of parameters: we introduce

$$\epsilon = \left(\frac{\nu}{\beta}\right)^{1/3}, \quad \epsilon_s = \frac{r}{2\beta}.$$

We suppose ν constant, but $r \gg 1$ such that $\varepsilon\varepsilon_s^{-1} \to 0$. Physically, this scaling expresses that we do not neglect the Ekman pumping anymore.

A priori, this system can not be treated as the previous ones, since there are two parameters, namely ε and ε_s. In order to overcome this difficulty, we link these parameters artificially: we assume that $\varepsilon = f(\varepsilon_s)$, with

$$\lim_{\varepsilon_s \to 0} \varepsilon\varepsilon_s^{-1} = 0.$$

With this assumption, symbol A depends on the parameter ε_s only. This shows that our method can be applied to problems including several parameters. The equation $a^{\varepsilon_s} = 0$ is equivalent to

$$\tau\xi^2 - i\nu\xi^4 - i\nu\varepsilon_s\varepsilon^{-3}\xi^2 - \nu\varepsilon^{-3}\xi_1 = 0.$$

For the sake of brevity, we do not detail computations here. There are two possible layers:

- a boundary layer with size ε_s, known as "Stommel Layer" (see [6]). If $\xi_2 \neq 0$, any corresponding boundary layer term satisfies

$$u_1 \sim C_1\varepsilon_s, \quad u_2 \sim C_2, \quad \varepsilon \to 0,$$

  and equations

$$\begin{aligned} \tilde{u}_2^0 + \nu\partial_X\tilde{u}_2^0 &= 0, \\ \partial_X\tilde{u}_1^0 + \partial_y\tilde{u}_2^0 &= 0. \end{aligned}$$

- a boundary layer with size $\varepsilon_s^{-1/2}\varepsilon^{3/2}$, known as "Friction layer". As above, if $\xi_2 \neq 0$, equations are

$$\begin{aligned} -\partial_X^3\tilde{u}_2^0 + \tilde{u}_2^0 &= 0, \\ \partial_X\tilde{u}_1^0 + \partial_y\tilde{u}_2^0 &= 0. \end{aligned}$$

  There again the terms we have neglected in the equations are negligible. All these results are in agreement with [20, 6].

7. Spherical cases

The case of spherical layers is central with regards to its applications in geophysics. In order to derive its main features, we may adapt considerations of section 2. Let us consider the case of a boundary layer at latitude θ. The natural directions of the problem are those tangent and perpendicular to the sphere, so that we make the change of variables

$$\mathbf{x}' = R_\theta\mathbf{x}, \quad v' = R_\theta v,$$

where R_θ is the rotation of angle θ in the (x, z) plane, v being any 3-D vectorial quantity in the equations (fluid velocity, current density, ...). This leads to a new linear system

$$A^{\varepsilon,\theta}U' = 0. \tag{7.1}$$

The main difference with the flat case is the geometrical constraint on the length scales brought by sphericity: if δ (resp. h) is the typical length scale of the layer perpendicularly (resp. tangently) to the sphere, we have $h = O(\sqrt{\delta})$. Following this, we look for solutions $U' = e^{i\tau^\varepsilon t} e^{i\xi^\varepsilon \mathbf{x}'} v^\varepsilon$ of (7.1) with τ, ξ_2 independent of ε, and

$$\mathcal{I}m\, \xi_1^\varepsilon \to +\infty,\ \mathcal{I}m\, \xi_3^\varepsilon \to +\infty,\ |\mathcal{I}m\, \xi_3^\varepsilon| \sim C|\mathcal{I}m\, \xi_1^\varepsilon|^{1/2}. \tag{7.2}$$

Note that ξ_1^ε corresponds to δ, and goes faster to infinity than ξ_3^ε.

7.1. Rotating fluids

After rotation of angle θ, we get the system

$$A^{\varepsilon,\theta} U' = 0, \quad U' = \begin{pmatrix} u' \\ p' \end{pmatrix},$$

$$A^{\varepsilon,\theta}(\tau, \xi) = \begin{pmatrix} i\tau + \varepsilon\xi^2 & -\varepsilon^{-1}\cos(\theta) & 0 & \varepsilon^{-1}\xi_1 \\ \varepsilon^{-1}\cos(\theta) & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\sin(\theta) & \varepsilon^{-1}\xi_2 \\ 0 & \varepsilon^{-1}\sin(\theta) & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}\xi_3 & 0 \end{pmatrix}.$$

Equation $a^{\varepsilon,\theta} = 0$ is

$$\varepsilon^2\xi^2(i\tau + \varepsilon\xi^2)^2 + \sin^2(\theta)\xi_1^2 + \cos^2(\theta)\xi_3^2 - 2\cos(\theta)\sin(\theta)\xi_1\xi_3 = 0. \tag{7.3}$$

We must distinguish two cases.

7.1.1. Far from the equator

We suppose θ constant different from zero. The leading terms in the left part of (7.3) are $\varepsilon^2\xi^2(\varepsilon\xi^2)^2$ and $\sin^2(\theta)\xi_1^2$, and we get:

$$\varepsilon^4\xi_1^6 = -\sin^2(\theta)\xi_1^2, \tag{7.4}$$

which gives the boundary layer size $\varepsilon^{1/2}(\sin\theta)^{-1/2}$. The features are essentially those of the Ekman layer, with elements $\begin{pmatrix} u' \\ p' \end{pmatrix}$ of $\ker(A^{\theta,\varepsilon})$ satisfying up to a multiplicative factor:

$$u_1' \sim C_1\varepsilon, \quad u_2' \sim C_2, u_3' \sim C_3, \quad p' \sim C_4\varepsilon^2, \quad \varepsilon \to 0.$$

The equations are (dropping the primes):

$$\begin{aligned} \partial_X^2 \tilde{u}_2^0 - \sin(\theta)\tilde{u}_1^0 &= 0, \\ \partial_X^2 \tilde{u}_1^0 + \sin(\theta)\tilde{u}_2^0 &= 0. \end{aligned}$$

7.1.2. Near the equator

When θ goes to zero, the previous size becomes incorrect as $\sin(\theta)$ goes to zero. This phenomenon is called "geostrophic degeneracy". We want to know the value of θ below which it is not valid anymore, and then to find the boundary layer size at the equator.

To answer the first question, we consider equation (7.3) for $\theta = \varepsilon^{\gamma}$. We want to determine the critical γ_c beyond which the boundary layer size changes. For $\gamma < \gamma_c$, (7.4) gives

$$\xi_1(\varepsilon) \sim C\varepsilon^{(\gamma-1)/2}, \quad \varepsilon \to 0,$$

and the terms in the left part of (7.3) are of respective sizes

$$\varepsilon^2\xi^2(i\tau + \varepsilon\xi^2)^2 = O\left(\varepsilon^{3\gamma-2}\right), \qquad \sin^2(\theta)\,\xi_1^2 = O\left(\varepsilon^{3\gamma-2}\right), \tag{7.5}$$

$$\cos^2(\theta)\,\xi_3^2 = O\left(\varepsilon^{\gamma/2-1}\right), \qquad 2\cos(\theta)\,\sin(\theta)\,\xi_1\,\xi_3 = O\left(\varepsilon^{7\gamma/4-3/2}\right). \tag{7.6}$$

At $\gamma = \gamma_c$, the leading terms of (7.3) must change, to provide a new boundary layer size. A change can happen iff:

$$3\gamma - 2 = \gamma/2 - 1 \quad \text{or} \quad 3\gamma - 2 = 7\gamma/4 - 3/2.$$

Both cases lead to $\gamma_c = 2/5$.

At the equator ($\theta = 0$) equation (7.3) is:

$$\varepsilon^2\xi^2(i\tau + \varepsilon\xi^2)^2 + \xi_3^2 = 0.$$

and we get the boundary layer size at the equator: $\varepsilon^{4/5}$.

Remark. The values we have found about equator degeneracy match the results of Stewartson in article [25].

Corresponding elements of $\ker(A^{\theta,\varepsilon})$ ($\theta = 0$) satisfy in that case

$$u_1' \sim C_1\varepsilon^{2/5}, \quad u_2' \sim C_2, u_3' \sim C_3, \quad p' \sim C_4\varepsilon^{4/5}, \quad \varepsilon \to 0.$$

We find equations (dropping the primes)

$$\begin{aligned} \tilde{u}_2^0 - \partial_X \tilde{p}^0 &= 0, \\ \tilde{u}_1^0 + \partial_X^2 \tilde{u}_2^0 &= 0, \\ \partial_z \tilde{p}^0 + \partial_X^2 \tilde{u}_3^0 &= 0, \\ \partial_X \tilde{u}_1^0 + \partial_z \tilde{u}_3^0 &= 0, \end{aligned}$$

which give the equation on pressure

$$\frac{\partial^2 \tilde{p}^0}{\partial z^2} + \frac{\partial^6 \tilde{p}^0}{\partial X^6} = 0.$$

Remark. The non-linear term has size

$$u \cdot \nabla u = (\varepsilon^{1/5}, \varepsilon^{-1/5}, \varepsilon^{-1/5})$$

and is small compared to the terms involved in the equations of the layer.

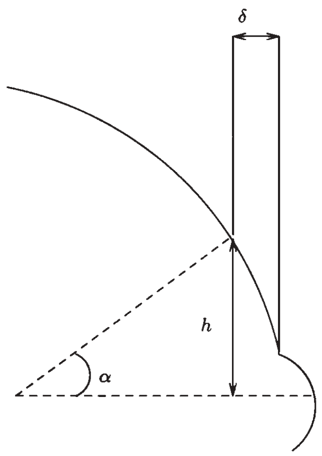

FIG. 2. Derivation of the $E^{4/7}$ boundary layer size.

7.1.3. Other layers

In [25], Stewartson considers the problem of two concentric spheres rotating at slightly different speeds. After huge calculations, he enumerates five types of layers: ε and $\varepsilon^{4/5}$ spherical layer, $\varepsilon^{2/3}$ and $\varepsilon^{1/2}$ cylindrical layers, plus a layer of size $\varepsilon^{4/7}$ located on the cylinder circumscribing the inner sphere. This last layer comes from the pumping due to the Ekman layers: see [20] or [12] for physical description of this phenomenon. To model the evolution of the fluid, appropriate equations are (far from the Ekman layers)

$$\partial_t u - \varepsilon \Delta u + \beta u + \nabla p = 0, \tag{7.7}$$

$$\nabla \cdot u = 0, \tag{7.8}$$

as they take into account both the Taylor–Proudman theorem (u is the horizontal velocity, the equation is 2-D) and the pumping of the (spherical) Ekman-type layer (through the βu term). Here, $\beta \sim C \sin(\alpha)^{-1/2}$. As $\varepsilon \to 0$, we get the relation

$$\varepsilon \xi_1^2 \sim C \sin(\alpha)^{-1/2}.$$

But geometrical constraints give $\sin(\alpha)^{-1} \sim \xi_1^{1/2}$. We get $\xi_1 \sim C\varepsilon^{-4/7}$, which is the size expected (see figure).

7.2. Hartmann case

We can lead the same analysis with system of section 4. We do not give any detail of the computations, analogous to the preceding ones.

– For θ constant $\neq 0$, we find a unique boundary layer size $\varepsilon^{1/2}$.

– As θ goes to zero, this size is not valid anymore. If we set $\theta = \varepsilon^\gamma$, we find that the "critical" γ is $\gamma_c = \varepsilon^{1/6}$. At the equator, the boundary layer size is $\varepsilon^{1/3}$.

Acknowledgements. I wish to thank Emmanuel Grenier for his many suggestions and advice.

References

[1] L. V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York 1978.

[2] A. Babin, A. Mahalov and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier–Stokes equations for uniformly rotating fluids, *European J. Mech. B. Fluids* **15** (3) (1996), 291–300.

[3] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Ekman boundary layers of rotating fluids*, to appear (2001).

[4] T. Colin and P. Fabrie, Rotating fluid at high Rossby number driven by a surface stress: existence and convergence, *Adv. Differential Equations*, FJOURNAL = Advances in Differential Equations **2** (5) (1997), 715–751.

[5] B. Desjardins, E. Dormy and E. Grenier, Stability of mixed Ekman–Hartmann boundary layers, *Nonlinearity*, Nonlinearity **12** (2) (1999), 181–199.

[6] B. Desjardins and E. Grenier, On the homogeneous model of wind-driven ocean circulation, *SIAM J. Appl. Math.* **60** (1) (2000), 43–60 (electronic).

[7] E. Dormy, *Modélisation Numérique de la Dynamo Terrestre*, PhD thesis, Institut de Physique du Globe de Paris, 1997.

[8] W. Eckhaus, *Asymptotic Analysis of Singular Perturbations*, North-Holland, Amsterdam, 1979.

[9] D. Gérard-Varet, A geometric optics type approach to fluid boundary layers, *Comm. Partial Differential Equations*, to appear.

[10] D. Gérard-Varet and E. Grenier, A zoology of boundary layers, *Rev. R. Acad. Cien. Serie A. Math.* **96** (2002), 401–410.

[11] M. Gisclon and D. Serre, Étude des conditions aux limites pour un système strictement hyberbolique via l'approximation parabolique, *C. R. Acad. Sci. Paris Sér. I Math.* **319** (4) (1994), 377–382.

[12] H. P. Greenspan, *The Theory of Rotating Fluids*, Breukelen Press, Brookline 1968.

[13] E. Grenier, Non dérivation des équations de Prandtl, in: *Séminaire sur les Équations aux Dérivées Partielles, 1997–1998*, École Polytech., Palaiseau 1998.

[14] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Comm. Partial Differential Equations* **22** (5–6) (1997), 953–975.

[15] O. Guès, TITLE = Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites, *Ann. Inst. Fourier (Grenoble)* **45** (4) (1995), 973–1006.

[16] J. Kevorkian and J. D. Cole, *Multiple scale and singular perturbation methods*, Springer-Verlag, Heildelberg, Berlin, New York 1996.

[17] L. Landau and E. Lifschitz, *Mécanique des Fluides*, Mir, Moscow 1959.

[18] R. J. Lingwood and T. Alboussiere, On the stability of the Hartmann layer, *Physics of Fluids* **11** (8) (1999), 2058–2068.

[19] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data, *Comm. Pure Appl. Math.* **53** (4) (2000), 432–483.

[20] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York 1979.

[21] J. Rauch and M. Keel, Lectures on geometric optics, in: *Hyperbolic equations and frequency interactions*, 5, IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI 1999.

[22] P. H. Roberts, *An introduction to magnetohydrodynamics*, Longmans, Green, London 1967.

[23] J. Sanchez-Hubert and E. Sanchez-Palencia, *Introduction aux méthodes asymptotiques et à l'homogénéisation*, Masson, Paris 1992.
[24] K. Stewartson, On almost rigid rotations, *J. Fluid Mech.* **3** (1957), 17–26.
[25] K. Stewartson, On almost rigid rotations. Part 2, *J. Fluid Mech.* **26** (1966), 131–152.
[26] A. I. Van De Vooren, The connection between Ekman and Stewartson layers for a rotating disk, **27** (1993), 189–207.
[27] M. Van Dyke, *Perturbation methods in fluid mechanics*, The Parabolic Press, Stanford 1975.
[28] R. Zeytounian, *Modélisation asymptotique en mécanique des fluides newtoniens*, Springer-Verlag, 1994.

David Gerard-Varet
U.M.P.A
U.M.R. 5669
E.N.S. Lyon
46 Allée d'Italie
69364 Lyon
France

(accepted: November 27, 2003)