

On the ill-posedness of the Prandtl equation

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Abstract

The concern of this paper is the Cauchy problem for the Prandtl equation. This problem is known to be well-posed for analytic data [13, 10], or for data with monotonicity properties [11, 15]. We prove here that it is linearly ill-posed in Sobolev type spaces. The key of the analysis is the construction, at high tangential frequencies, of unstable quasimodes for the linearization around solutions with non-degenerate critical points. Interestingly, the strong instability is due to viscosity, which is coherent with well-posedness results obtained for the inviscid version of the equation [8]. A numerical study of this instability is also provided.

1 Introduction

One challenging open problem of fluid dynamics is to understand the inviscid limit of the Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u}^\nu + \mathbf{u}^\nu \cdot \nabla \mathbf{u}^\nu + \nabla p^\nu - \nu \Delta \mathbf{u}^\nu = 0, & \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{u}^\nu = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u}^\nu|_{\partial\Omega} = 0, \end{cases} \quad (1.1)$$

in a domain Ω *with boundaries, endowed with a no-slip boundary condition*. Mathematically, the main difficulty is the lack of uniform bounds on the vorticity field, as the viscosity ν goes to zero. In terms of fluid dynamics, this corresponds to a boundary layer phenomenon near $\partial\Omega$.

A natural approach to describe this boundary layer is to look for a double-scale asymptotics, with a parabolic scaling in the normal direction. Consider the case $\Omega \subset \mathbb{R}^2$. At least locally, any point \mathbf{x} in a neighborhood of $\partial\Omega$ has a unique decomposition

$$\mathbf{x} = y \mathbf{n}(x) + \tilde{\mathbf{x}}(x), \quad \tilde{\mathbf{x}} \in \partial\Omega,$$

where $y > 0$, x is an arc length parametrization of the boundary, and \mathbf{n} is the inward unit normal vector at $\partial\Omega$. The velocity field can be written

$$\mathbf{u}^\nu(t, \mathbf{x}) = u^\nu(t, x, y) \boldsymbol{\tau}(x) + v^\nu(t, x, y) \mathbf{n}(x),$$

where $(\boldsymbol{\tau}, \mathbf{n})$ is the Frénet frame. It is then natural to consider an approximation of the type:

$$\begin{aligned} u^\nu(t, x, y) &\approx u^0(t, x, y) + u^{BL}(t, x, y/\sqrt{\nu}), \\ v^\nu(t, x, y) &\approx v^0(t, x, y) + \sqrt{\nu} v^{BL}(t, x, y/\sqrt{\nu}), \end{aligned} \quad (1.2)$$

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where

$$\mathbf{u}^0(t, x, y) = u^0(t, x, y) \boldsymbol{\tau}(x) + v^0(t, x, y) \mathbf{n}(x)$$

satisfies the Euler equation with the no penetration condition, and

$$(u^{BL}, v^{BL}) = (u^{BL}, v^{BL})(t, x, Y)$$

describes a boundary layer corrector with typical scale $\sqrt{\nu}$ in the normal direction. It is slightly more convenient to introduce

$$u(t, x, Y) := u^0(t, x, 0) + u^{BL}(t, x, Y), \quad v(t, x, Y) := Y \partial_y v^0(t, x, 0) + v^{BL}(t, x, Y).$$

Indeed, inserting this Ansatz in the Navier-Stokes equations, we get formally

$$\left\{ \begin{array}{l} \partial_t u + u \partial_x u + v \partial_Y u - \partial_Y^2 u = (\partial_t u^0 + u^0 \partial_x u^0)|_{y=0}, \quad Y > 0, \\ \partial_x u + \partial_Y v = 0, \quad Y > 0, \\ (u, v) = 0, \quad Y = 0, \\ \lim_{Y \rightarrow +\infty} u = u^0|_{y=0}. \end{array} \right. \quad (1.3)$$

These are the so-called Prandtl equations, derived by Ludwig Prandtl [12]. Note that the curvature of the boundary does not appear explicitly in the system. It is however involved in (1.3) through the Euler field, and through the interval of definition of the arc length parametrization x . Up to our knowledge, all studies deal with one of the three following cases: $x \in \mathbb{R}$, $x \in \mathbb{T}$, or $0 < x < L$, supplemented with a condition on u at $x = 0$. The first and second choices are convenient to describe phenomena that are local in x . The case $x \in \mathbb{T}$ may also model the outside of a bounded convex obstacle. Finally, the third configuration is adapted to the spreading of a flow around a thin obstacle, where $x = 0$ corresponds to the tip of the obstacle.

Although this formal asymptotics is very natural, its validity is not clear. As emphasized by physicists, including Prandtl himself, it may not hold uniformly in space and time. One reason is the so-called *boundary layer separation*, which is observed for flows around obstacles, see [7]. Nevertheless, the description (1.2) fits with many experiments, upstream from the separation zone. In any case, to understand the relevance and limitations of the Prandtl model is a crucial issue.

From the mathematical point of view, one must address two problems:

1. The well-posedness of the Prandtl equation.
2. The justification of the expansion (1.2).

These two problems depend crucially on the choice of the underlying functional spaces, especially on the regularity that is required in the tangential variable x . Indeed, the main mathematical difficulty is the lack of control of the x derivatives. For example, v is recovered in (1.3) through the divergence condition, and in terms of x -regularity, behaves broadly like $\partial_x u$. This loss of one derivative is not balanced by any horizontal diffusion term, so that standard energy estimates do not apply.

Within spaces of functions that are analytic in $x \in \mathbb{R}$, $Y \in \mathbb{R}^+$, Sammartino and Caffish have overcome these problems, justifying locally in time the boundary layer asymptotics

[13, 14]. But for more “realistic” functional settings, the way solutions of (1.1) behave is still poorly understood. Various instability mechanisms, that are filtered out in an analytic framework, become a huge source of trouble. For example, when the viscosity is small, the Navier-Stokes equation admits exponentially growing solutions which are both small-scale and isotropic in x, y . Their evolution is lost in the anisotropic Prandtl description.

This remark was used by Emmanuel Grenier in [6], who relied on the so-called Rayleigh instability for inviscid flows to show that *the asymptotics (1.2) does not (always) hold in the Sobolev space H^1* (see [6] for a precise statement). However, the relevance of this asymptotics in L^p , or its relevance in the absence of Rayleigh instabilities, are still open issues.

Above all, the local in time well-posedness of the Prandtl equation for smooth (say Sobolev) initial data has been so far an open problem. Up to our knowledge, the Cauchy problem has only been solved in two settings:

- i) $x \in \mathbb{R}$, with data that are analytic in x : see [13, 10] for more.
- ii) $0 < x < L$, with data that are monotonic in y : see [11, 15] for more.

One may also cite article [4], in which blow up in time of some smooth solutions is exhibited. Finally, let us mention the interesting work [8], in which the inviscid version of (1.3) is analyzed (no $\partial_y^2 u$ in the equation). Interestingly, for a smooth initial data, this equation turns out to have an explicit solution through the method of characteristics. In particular, starting from a smooth data, one recovers locally in time a smooth data. More precisely, there is only a finite loss of x -derivatives, so that the Cauchy problem is (weakly) well-posed. We refer to [8] for all details. See also papers [5, 1] on the hydrostatic equations, that share some features with Prandtl equations. For more on Prandtl equations, see the review [3].

On the basis of the inviscid result, it seems reasonable to bet for well-posedness of the Prandtl equation (1.3) in Sobolev type spaces. *The aim of this paper is to show that it is actually linearly ill-posed in this framework.* As we shall see later on, the reason for ill-posedness is a strong destabilization mechanism due to two ingredients: viscosity, and critical points in the base velocity profile. In particular, it does not contradict the positive results obtained in the inviscid case and for monotonic data.

We now describe our results. We consider the case $(x, Y) \in \mathbb{T} \times \mathbb{R}^+$ and $\mathbf{u}^0(t, x, 0) = 0$. Up to minor changes, we could also consider Euler fields satisfying $\mathbf{u}^0(t, x, 0) = (cst, 0)$, which has more physical meaning. To lighten notations, we write y instead of Y . The Prandtl equation comes down to

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u, v)|_{y=0} = (0, 0), & \lim_{y \rightarrow +\infty} u = 0. \end{cases} \quad (1.4)$$

Let $u_s = u_s(t, y)$ a smooth solution of the heat equation

$$\partial_t u_s - \partial_y^2 u_s = 0, \quad u_s|_{y=0} = 0, \quad u_s|_{t=0} = U_s, \quad (1.5)$$

with good decay as $y \rightarrow +\infty$. Clearly, the shear velocity profile $(u_s, v_s) = (u_s(t, y), 0)$ satisfies

the system (1.4). We consider the linearization around (u_s, v_s) , that is

$$\begin{cases} \partial_t u + u_s \partial_x u + v \partial_y u_s - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u, v)|_{y=0} = (0, 0), & \lim_{y \rightarrow +\infty} u = 0. \end{cases} \quad (1.6)$$

We wish to study well-posedness properties of (1.6), for a certain class of velocities u_s . In this view, we introduce the following functional spaces:

$$W_\alpha^{s,\infty}(\mathbb{R}^+) := \{f = f(y), \quad e^{\alpha y} f \in W^{s,\infty}(\mathbb{R}^+)\}, \quad \forall \alpha, s \geq 0,$$

with $\|f\|_{W_\alpha^{s,\infty}} := \|e^{\alpha y} f\|_{W^{s,\infty}}$, and

$$E_{\alpha,\beta} := \left\{ u = u(x, y) = \sum_{k \in \mathbb{Z}} \hat{u}^k(y) e^{ikx}, \quad \|\hat{u}^k\|_{W_\alpha^{0,\infty}} \leq C_{\alpha,\beta} e^{-\beta|k|}, \forall k \right\}, \quad \forall \alpha, \beta > 0,$$

with $\|u\|_{E_{\alpha,\beta}} := \sup_k e^{\beta|k|} \|\hat{u}^k\|_{W_\alpha^{0,\infty}}$.

Note that the functions of $E_{\alpha,\beta}$ have analytic regularity in x . They have only L^∞ regularity in y , with an exponential weight. More regularity in y could be considered as well. Let $\alpha, \beta > 0$. We prove in the appendix the following result:

Proposition 1 (Well-posedness in the analytic setting)

Let $u_s \in C^0(\mathbb{R}_+; W_\alpha^{1,\infty}(\mathbb{R}_+))$. There exists $\rho > 0$ such that: for all T with $\beta - \rho T > 0$, and all $u_0 \in E_{\alpha,\beta}$, the linear equation (1.6) has a unique solution

$$u \in C([0, T]; E_{\alpha,\beta-\rho T}), \quad u(t, \cdot) \in E_{\alpha,\beta-\rho t}, \quad u|_{t=0} = u_0.$$

In short, the Cauchy problem for (1.6) is locally well-posed in the analytic setting. We shall denote

$$T(t, s)u_0 := u(t, \cdot)$$

where u is the solution of (1.6) with $u|_{t=s} = u_0$. As the spaces $E_{\alpha,\beta}$ are dense in the spaces

$$H^m := H^m(\mathbb{T}_x, W_\alpha^{0,\infty}(\mathbb{R}_y^+)), \quad m \geq 0,$$

this makes sense to introduce the following notation: for all $T \in \mathcal{L}(E_{\alpha,\beta}, E_{\alpha,\beta'})$,

$$\|T\|_{\mathcal{L}(H^{m_1}, H^{m_2})} = \sup_{u_0 \in E_{\alpha,\beta}} \frac{\|Tu_0\|_{H^{m_2}}}{\|u_0\|_{H^{m_1}}},$$

that belongs to $\mathbb{R}_+ \cup \{+\infty\}$. In particular, it is infinite when T does not extend to a bounded operator from H^{m_1} to H^{m_2} . The main result of our paper is

Theorem 1 (Ill-posedness in the Sobolev setting)

i) Let $u_s \in C^0(\mathbb{R}_+; W_\alpha^{4,\infty}(\mathbb{R}_+)) \cap C^1(\mathbb{R}_+; W_\alpha^{2,\infty}(\mathbb{R}_+))$. Assume that the initial velocity has a non-degenerate critical point over \mathbb{R}^+ . Then, there exists $\sigma > 0$, such that for all $\delta > 0$,

$$\sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t, s)\|_{\mathcal{L}(H^m, H^{m-\mu})} = +\infty, \quad \forall m \geq 0, \mu \in [0, 1/2].$$

ii) Moreover, one can find solutions u_s of (1.5) and $\sigma > 0$ such that: for all $\delta > 0$,

$$\sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t, s)\|_{\mathcal{L}(H^{m_1}, H^{m_2})} = +\infty, \quad \forall m_1, m_2 \geq 0.$$

This theorem expresses strong linear ill-posedness of the Prandtl equation in the Sobolev framework. It is a consequence of an instability process, that holds at high tangential frequencies. We will show that some perturbations with tangential frequency $k \gg 1$ grow in the linear régime like $e^{\sqrt{kt}}$. The extension of our instability result to the nonlinear setting is still an open issue. In particular, we are not able to apply Grenier's method to go from linear to nonlinear instability: indeed, it requires upper bounds on $T(t, s)$ that we do not have. However, we believe this difficulty to be purely technical: in our view, it is very unlikely that this destabilization phenomenon can be cancelled by nonlinear interactions.

The outline of the paper is as follows. Section 2 gives a formal description of the instability mechanism. It relies on an asymptotic analysis of (1.6), in the high tangential frequency limit. Thanks to this analysis, we show that ill-posedness for the PDE (1.6) comes down to a “spectral condition” for a reduced ODE, namely:

(SC) There exists $\tau \in \mathbb{C}$ with $\text{Im } \tau < 0$, and a solution $W = W(z)$ of

$$(\tau - z^2)^2 \frac{d}{dz} W + i \frac{d^3}{dz^3} ((\tau - z^2)W) = 0, \quad (1.7)$$

such that $\lim_{z \rightarrow -\infty} W = 0$, $\lim_{z \rightarrow +\infty} W = 1$.

This spectral condition is studied in section 3, and shown to be satisfied. On these grounds, we prove theorem 1, cf section 4. We end up the paper with numerical computations, which emphasize that our instability mechanism is effective.

2 The instability mechanism

In this section, we describe the destabilization of system (1.6), leading to the ill-posedness theorem. As we shall see, it takes place at high tangential frequencies, say $O(1/\varepsilon)$, and has a typical time $O(\sqrt{\varepsilon})$. At this timescale, the time dependence of the base velocity $(u_s(t, y), 0)$ will not play an important role. Thus, to understand the instability mechanism, we can consider the simpler equation

$$\begin{cases} \partial_t u + U_s \partial_x u + v U_s' - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u, v)|_{y=0} = (0, 0). \end{cases} \quad (2.1)$$

Handling of the real equation, that is with u_s instead of U_s , will require minor modifications, to be achieved in section 4.

System (1.6) has constant coefficients in t and y , so that we can perform a Fourier analysis: we look for solutions in the form

$$u(t, y) = e^{ik(\omega(k)t+x)} \hat{u}^k(y), \quad v = k e^{ik(\omega(k)t+x)} \hat{v}^k(y), \quad k > 0. \quad (2.2)$$

As we are interested in high tangential frequencies, we denote $\varepsilon := 1/k \ll 1$, and write $\omega(\varepsilon)$ instead of $\omega(k)$, $u_\varepsilon(y), v_\varepsilon(y)$ instead of $\hat{u}^k(y), \hat{v}^k(y)$. The divergence condition yields $v'_\varepsilon(y) = -iu_\varepsilon(y)$. Using this relation in the first equation in (2.1), one ends up with

$$\begin{cases} (\omega(\varepsilon) + U_s) v'_\varepsilon - U'_s v_\varepsilon + i\varepsilon v_\varepsilon^{(3)} = 0, & y > 0, \\ v_\varepsilon|_{y=0} = v'_\varepsilon|_{y=0} = 0. \end{cases} \quad (2.3)$$

Thus, the high frequency limit $\varepsilon \rightarrow 0$ in variable x yields a singular perturbation problem in variable y . To investigate this problem, one must first consider the inviscid case $\varepsilon = 0$.

2.1 The inviscid case

When $\varepsilon = 0$, one can *a priori* only retain the impermeability condition. The appropriate problem is

$$(\omega + U_s) v' - U'_s v = 0, \quad y > 0, \quad v|_{y=0} = 0. \quad (2.4)$$

This spectral problem, as well as the corresponding evolution equation, have been studied exhaustively in [8]. Clearly, there are non-trivial solutions if and only if ω belongs to the range of $-U_s$. Moreover, the couples

$$\omega_a = -U_s(a), \quad v_a = H(y - a) (U_s - U_s(a)), \quad a > 0$$

where H is the Heaviside function, satisfy (2.4). Note that the regularity of v_a depends on the choice of a . When a is a critical point, it belongs to $W_\alpha^{2,\infty}(\mathbb{R}^+)$ with a discontinuous second derivative. Otherwise, it is only in $W_\alpha^{1,\infty}(\mathbb{R}^+)$, with a discontinuous first derivative. Luckily enough, the additional boundary condition $v'_a|_{y=0} = 0$ is also satisfied.

2.2 The viscous perturbation

When ε is not 0, the inviscid eigenelements ω_a, v_a do not solve (2.3). All boundary conditions are satisfied, *cf* the above remark, but the equation is not. First, there is a $O(\varepsilon)$ remaining term for $y > a$. More importantly, v_a is not smooth at $y = a$, whereas a solution of this parabolic equation should be.

Nevertheless, at least if a is a non-degenerate critical point, there is an approximate solution near (ω_a, v_a) . We shall establish this rigorously in section 4. We just give here a formal expansion. It reads

$$\begin{cases} \omega(\varepsilon) \sim \omega_a + \varepsilon^{1/2} \tau, \\ v_\varepsilon(y) \sim v_a + \varepsilon^{1/2} \tau H(y - a) + \varepsilon^{1/2} V\left(\frac{y - a}{\varepsilon^{1/4}}\right), \end{cases} \quad (2.5)$$

where $\tau \in \mathbb{C}$, and $V = V(z)$ quickly tends to zero as $z \rightarrow \pm\infty$. Note that the approximation of v_ε has two parts: the “regular” part

$$v_\varepsilon^{reg}(y) = H(y - a) \left(U_s(y) - U_s(a) + \varepsilon^{1/2} \tau \right)$$

and the “shear layer part”

$$v_\varepsilon^{sl}(y) = \varepsilon^{1/2} V\left(\frac{y - a}{\varepsilon^{1/4}}\right).$$

For $\omega(\varepsilon) = -U_s(a) + \varepsilon^{1/2}\tau$, the function v_ε^{reg} solves (2.3) up to $O(\varepsilon)$, away from the critical point $y = a$. However, it has a jump at $y = a$, together with its second derivative. The role of the shear layer v_ε^{sl} , which concentrates near $y = a$, is to cancel these discontinuities. Still formally, we obtain the system satisfied by the profile V :

$$\begin{cases} \left(\tau + U_s''(a) \frac{z^2}{2} \right) V' - U_s''(a) z V + i V^{(3)} = 0, & z \neq 0, \\ [V]_{|z=0} = -\tau, \quad [V']_{|z=0} = 0, \quad [V'']_{|z=0} = -U_s''(a), \\ \lim_{\pm\infty} V = 0. \end{cases} \quad (2.6)$$

Let us point out that this system is *a priori* overdetermined, as jump and boundary conditions provide too many constraints. This justifies the introduction of the parameter τ in the Ansatz (2.5). As we shall see below, there is a τ for which system (2.6) has a solution. *Moreover, $\text{Im} \tau$ is negative. Hence, back to the Fourier representation (2.2), the k -th mode will grow in time like $e^{\sqrt{k}t}$. This is the key of the instability mechanism.*

To see how the condition (SC) of the introduction steps in, we need a few rewritings. First, $\tau + U_s''(a) \frac{z^2}{2}$ satisfies the equation in (2.6). We therefore introduce

$$\tilde{V}(z) = V(z) + \mathbf{1}_{\mathbb{R}_+} \left(\tau + U_s''(a) \frac{z^2}{2} \right)$$

which leads to

$$\begin{cases} \left(\tau + U_s''(a) \frac{z^2}{2} \right) \tilde{V}' - U_s''(a) z \tilde{V} + i \tilde{V}^{(3)} = 0, & z \in \mathbb{R}, \\ \lim_{-\infty} \tilde{V} = 0, \quad \tilde{V} \sim_{+\infty} \tau + U_s''(a) \frac{z^2}{2}. \end{cases}$$

Then, we introduce W such that

$$\tilde{V} = \left(\tau + U_s''(a) z^2/2 \right) W.$$

We get:

$$\begin{cases} \left(\tau + U_s''(a) z^2/2 \right)^2 \frac{d}{dz} W + i \frac{d^3}{dz^3} \left(\left(\tau + U_s''(a) z^2/2 \right) W \right) = 0, \\ \lim_{-\infty} W = 0, \quad \lim_{+\infty} W = 1. \end{cases}$$

Finally, we perform the change of variables

$$\tau = \frac{1}{\sqrt{2}} |U_s''(a)|^{1/2} \tilde{\tau}, \quad z = 2^{1/4} |U_s''(a)|^{-1/4} \tilde{z}.$$

Dropping the tildes leaves us with the reduced ODE

$$\begin{cases} \left(\tau + \text{sign}(U_s''(a)) z^2 \right)^2 \frac{d}{dz} W + i \frac{d^3}{dz^3} \left(\left(\tau + \text{sign}(U_s''(a)) z^2 \right) W \right) = 0, \\ \lim_{-\infty} W = 0, \quad \lim_{+\infty} W = 1. \end{cases}$$

If $U_s''(a) < 0$, it is exactly the system in (SC). If on the contrary $U_s''(a) > 0$, and if (τ, W) satisfies the system in (SC), then $(\tau := -\bar{\tau}, W := \bar{W})$ satisfies the above system. In both

cases, back to the original system (2.6), condition (SC) gives a solution (τ, V) with $\mathcal{I}m \tau < 0$. In particular, this $\sqrt{\varepsilon}$ correction to the eigenvalue is a source of strong instability, leading to ill-posedness.

The proof of Theorem 1, which is based on this formal shear layer phenomenon, is postponed to section 4. In the next paragraph, we focus on condition (SC), and prove that it is satisfied.

3 The spectral condition (SC)

We need to study the existence of heteroclinic orbits for the ODE (1.7). Note that $W = 1$ is a solution. Equation (1.7) can be written as a second order equation in $X = W'$:

$$i(\tau - z^2)X'' - 6izX' + ((\tau - z^2)^2 - 6i)X = 0. \quad (3.1)$$

To show that (SC) holds, we proceed in three steps.

Step 1. We consider an auxiliary eigenvalue problem:

$$Au := \frac{1}{z^2 + 1}u'' + \frac{6z}{(z^2 + 1)^2}u' + \frac{6}{(z^2 + 1)^2}u = \alpha u. \quad (3.2)$$

For its study, we introduce the weighted spaces

$$\begin{aligned} \mathcal{L}^2 &:= \left\{ u \in L^2_{loc}, \int_{\mathbb{R}} (z^2 + 1)^4 |u|^2 < +\infty \right\}, \\ \mathcal{H}^1 &:= \left\{ u \in H^1_{loc}, \int_{\mathbb{R}} (z^2 + 1)^4 |u|^2 + \int_{\mathbb{R}} (z^2 + 1)^3 |u'|^2 < +\infty \right\} \hookrightarrow \mathcal{L}^2. \end{aligned}$$

with their obvious Hilbert norms. We see A as an operator from $D(A) := \{u \in \mathcal{H}^1, Au \in \mathcal{L}^2\}$ into \mathcal{L}^2 . *Our goal is to show that A has a positive eigenvalue.*

By standard arguments, the domain $D(A)$ is dense in \mathcal{L}^2 . Moreover, for any u in $D(A)$, there is a sequence u_n of smooth functions with compact support, such that $u_n \rightarrow u$ in \mathcal{H}^1 and $Au_n \rightarrow Au$ in \mathcal{L}^2 . Integration by parts and use of this density property give easily that A is symmetric, *i.e.*

$$\forall u, v \in D(A), \quad (Au | v)_{\mathcal{L}^2} = (Av | u)_{\mathcal{L}^2} \quad (3.3)$$

and that for λ large enough

$$((\lambda - A)u | u)_{\mathcal{L}^2} = \lambda \int_{\mathbb{R}} (z^2 + 1)^4 |u|^2 - 6 \int_{\mathbb{R}} (z^2 + 1)^2 |u|^2 + \int_{\mathbb{R}} (z^2 + 1)^3 |u'|^2 \geq \frac{1}{2} \|u\|_{\mathcal{H}^1}^2. \quad (3.4)$$

Then, the coercivity condition (3.4) allows to apply the Lax-Milgram lemma. It implies the invertibility of $\lambda - A$, with

$$\|(\lambda - A)^{-1} f\|_{\mathcal{L}^2} \leq \|(\lambda - A)^{-1} f\|_{\mathcal{H}^1} \leq C \|f\|_{\mathcal{L}^2}.$$

Moreover, from (3.3), $(\lambda - A)^{-1}$ is selfadjoint, and so is A .

First, let us prove that A has positive spectrum. To do so, we claim that it is enough to find $u \in D(A)$ with $(Au | u)_{\mathcal{L}^2} > 0$. Indeed, suppose *a contrario* that $\sigma(A)$ is contained in \mathbb{R}_- . Then, by the spectral theorem,

$$\forall \alpha > 0, \quad \|(A - \alpha)^{-1}\| = \frac{1}{d(\alpha, \sigma(A))} \leq \alpha^{-1}.$$

We deduce: for all $u \in D(A)$

$$\|u\|_{\mathcal{L}^2}^2 \leq \alpha^{-2} \|(A - \alpha)u\|_{\mathcal{L}^2}^2$$

Expanding the scalar products, we obtain

$$0 \leq \alpha^{-2} \|Au\|_{\mathcal{L}^2}^2 - 2\alpha^{-1} (Au | u)_{\mathcal{L}^2}$$

In the limit $\alpha \rightarrow +\infty$, we get $(Au | u)_{\mathcal{L}^2} \leq 0$ for all $u \in D(A)$. This proves our claim. From there, we simply take $u = e^{-2z^2}$. A straightforward computation gives

$$(Au | u) = \frac{439}{512} \sqrt{\pi} > 0,$$

and so $\sigma(A)$ has a positive subset.

It remains to exhibit a positive eigenvalue inside this positive subset of the spectrum. We remark that the operator A can be split into

$$A = A_1 + A_2, \quad A_1 u := \frac{1}{z^2 + 1} u'' + \frac{6z}{(z^2 + 1)^2} u', \quad A_2 u := \frac{6}{(z^2 + 1)^2} u.$$

On one hand, the operator A_1 is negative, and by Lax-Milgram Lemma, $A_1 - \lambda$ is invertible for any $\lambda > 0$. Thus, $\sigma(A_1) \subset \mathbb{R}_-$. On the other hand, let u_n a sequence with u_n and $A_1 u_n$ bounded in \mathcal{L}^2 . This implies that u_n is bounded in \mathcal{H}^1 , and so has a convergent subsequence in L^2_{loc} . Moreover, $|u_n|^2$ is equi-integrable over \mathbb{R} . Finally, it implies that $A_2 u_n$ has a convergent subsequence in \mathcal{L}^2 , which means that A_2 is A_1 -compact. Hence, the essential spectra of A and A_1 are the same, see [9]. In particular, the positive part of $\sigma(A)$ is made of isolated eigenvalues with finite multiplicity. Eventually, we state: *there exists $\alpha > 0$, and u in $D(A)$ satisfying (3.2).*

Step 2. We wish to convert the eigenlements (α, u) of the previous step into an appropriate solution (τ, X) of (3.1). We set $\tilde{\tau} = -\alpha^{1/2}$, and $\tilde{z} = \alpha^{-1/4} z$, $Y(\tilde{z}) = u(z)$. Dropping the tildes, we obtain a solution of

$$(\tau - z^2)Y'' - 6zY' + ((\tau - z^2)^2 - 6)Y = 0. \quad (3.5)$$

By a classical bootstrap argument, Y is smooth. Moreover, it inherits from u its integrability properties at infinity. Actually, the behaviour of Y can be further specified, as shown in:

Proposition 2 *The function Y admits a unique extension, still denoted by Y , that is holomorphic in z and satisfies (3.5) in the simply connected domain*

$$U_\tau := \mathbb{C} \setminus \left(\left[-i\infty, -i|\tau|^{1/2} \right] \cup \left[i|\tau|^{1/2}, +i\infty \right] \right).$$

Moreover, in the sectors $\arg z \in (-\pi/4 + \delta, \pi/4 - \delta)$ and $\arg z \in (3\pi/4 + \delta, 5\pi/4 - \delta)$, $\delta > 0$, it satisfies the inequality

$$|Y(z)| \leq C_\delta \exp(-z^2/4).$$

Proof. This proposition follows from the general theory of ODE's with holomorphic coefficients. The existence of a holomorphic solution is well-known, because the coefficient $\tau - z^2$ does not vanish on U_τ . As regards the inequality, we rewrite equation (3.1) as the first order system:

$$\frac{d}{dz}\mathcal{Y} = z\mathcal{A}(z)\mathcal{Y}, \quad \mathcal{Y} = \left(z^{-1}Y \quad \frac{d}{dz}Y \right), \quad \mathcal{A}(z) = \begin{pmatrix} 0 & 1 \\ \frac{6-(\tau-z^2)^2}{z^2(\tau-z^2)} & \frac{6}{\tau-z^2} - \frac{1}{z^2} \end{pmatrix}. \quad (3.6)$$

In particular, \mathcal{A} is holomorphic at infinity, with $\mathcal{A}(\infty) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It has two distinct eigenvalues ± 1 , with eigenvectors $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$.

Hence, we can apply [2, Theorem 5.1 p163]: in any closed sector S inside which $\operatorname{Re} z^2$ does not cancel, there exists solutions \mathcal{Y}_\pm (depending *a priori* on S) with the following asymptotic behaviour as $|z| \rightarrow +\infty$:

$$\mathcal{Y}_\pm \sim \left(\sum_{i \geq 0} \mathcal{Y}_\pm^i z^{\alpha_\pm - i} \right) e^{P_\pm(z)}, \quad (3.7)$$

where α_\pm is a complex constant, and $P_\pm(z)$ is a polynomial of degree 2. Moreover, the leading term of P_\pm is $\pm \frac{z^2}{2}$. Following the same scheme of proof, we get in the present case:

$$\mathcal{Y}_\pm^0 = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad \alpha_\pm = -\frac{1}{2}(\pm\tau + 7), \quad P_\pm = \pm z^2/2. \quad (3.8)$$

As our solution Y is integrable over \mathbb{R} , it is necessarily proportional to the decaying solution. The bounds in proposition 2 follow.

Now, as Y is defined on U_τ , we can perform the complex change of variable:

$$\tilde{z} := e^{i\pi/8}z, \quad \tilde{\tau} := e^{i\pi/4}\tau$$

Note that $\tilde{\tau}$ has a negative imaginary part. Moreover, for \tilde{z} real, the original variable z belongs to the sectors $\arg z \in (-\pi/4 + \delta, \pi/4 - \delta)$ or $\arg z \in (3\pi/4 + \delta, 5\pi/4 - \delta)$, with $\delta = 1/16$. By the proposition 2, the function $X(\tilde{z}) = Y(z)$ satisfies the estimate $|X(\tilde{z})| \leq C \exp(-1/4 \tilde{z}^2)$. Finally, dropping the tildes yields a solution τ, X of (3.1), where X decays at infinity. This concludes step 1.

Step 3. To deduce from the previous step that (SC) holds, it is enough that $\int_{\mathbb{R}} X(z)dz$ be non zero. If so, one can consider

$$W(z) := \left(\int_{\mathbb{R}} X(z')dz' \right)^{-1} \int_{-\infty}^z X(z') dz',$$

which clearly satisfies all requirements.

Let us assume *a contrario* that $\int_{\mathbb{R}} X(z)dz = 0$. Then, the function

$$V(z) := (\tau - z^2) \int_{-\infty}^z X(z') dz'$$

is a solution of

$$(\tau - z^2) V' + 2zV + iV^{(3)} = 0,$$

which decays exponentially as z goes to $\pm\infty$, together with all its derivatives. Differentiation of the equation gives

$$(\tau - z^2)V'' + 2V + iV^{(4)} = 0,$$

Then, we multiply by $\overline{V''}$, that is the complex conjugate of V'' , and integrate over \mathbb{R} . Simple integrations by parts yield:

$$\int_{\mathbb{R}} (\tau - z^2)|V''|^2 - 2 \int_{\mathbb{R}} |V'|^2 - i \int_{\mathbb{R}} |V^{(3)}|^2 = 0.$$

The imaginary part of this identity yields

$$\mathcal{I}m \tau \int_{\mathbb{R}} |V''|^2 = \int_{\mathbb{R}} |V^{(3)}|^2$$

which contradicts the fact that $\mathcal{I}m \tau < 0$. Thus, the condition (SC) is satisfied.

4 Proof of ill-posedness

Theorem 1 will be deduced from the formal analysis of section 2. This analysis was performed on (2.1), in which possible time variations of u_s were neglected. To account for the original system (1.6) will require a few modifications, notably in the choice of the approximation (2.5). We will distinguish between the parts i) and ii) of the theorem.

4.1 Ill-posedness for general u_s

Let u_s satisfying the assumptions of part i). Let a be the non-degenerate critical point of $u_s|_{t=0} = U_s$. For the sake of brevity, we consider the case $U''_s(a) < 0$, the other one being strictly similar. The differential equation

$$\partial_t \partial_y u_s(t, a(t)) + \partial_y^2 u_s(t, a(t)) a'(t) = 0, \quad a(0) = a$$

defines for small time $t < t_0$ a non-degenerate critical point $a(t)$ of $u_s(t, \cdot)$. Let then τ, W given by condition (SC). We set

$$V := (\tau - z^2)W - \mathbf{1}_{\mathbb{R}_+} (\tau - z^2).$$

In the light of section 2, we introduce, for $\varepsilon > 0$ and $t < t_0$:

$$\omega(\varepsilon, t) := -u_s(t, a(t)) + \frac{\varepsilon^{1/2}}{\sqrt{2}} |\partial_y^2 u_s(t, a(t))|^{1/2} \tau$$

as well as the “regular” velocity

$$v_\varepsilon^{reg}(t, y) := H(y - a(t)) \left(u_s(t, y) - u_s(t, a(t)) + \frac{\varepsilon^{1/2}}{\sqrt{2}} |\partial_y^2 u_s(t, a(t))|^{1/2} \tau \right),$$

and the shear layer velocity

$$v_\varepsilon^{sl}(t, y) := \frac{\varepsilon^{1/2}}{\sqrt{2}} \varphi(y - a(t)) |\partial_y^2 u_s(t, a(t))|^{1/2} V \left(|\partial_y^2 u_s(t, a(t))|^{1/4} \frac{(y - a(t))}{(2\varepsilon)^{1/4}} \right),$$

where φ is a smooth truncation function near 0. We then consider the following velocity field:

$$\begin{aligned} u_\varepsilon(t, x, y) &:= e^{i\varepsilon^{-1}x} U_\varepsilon(t, y), & U_\varepsilon(t, y) &= i e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_y \left(v_\varepsilon^{reg}(t, y) + v_\varepsilon^{sl}(t, y) \right), \\ v_\varepsilon(t, x, y) &:= e^{i\varepsilon^{-1}x} V_\varepsilon(t, y), & V_\varepsilon(t, y) &= \varepsilon^{-1} e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \left(v_\varepsilon^{reg}(t, y) + v_\varepsilon^{sl}(t, y) \right). \end{aligned}$$

In order to have a field that is 2π -periodic in x and growing in time, we take $\varepsilon := \frac{1}{n}$, with $n \in \mathbb{N}_*$. One verifies easily that $u_\varepsilon = e^{i\varepsilon^{-1}x} U_\varepsilon(t, y)$ is analytic in x , and $W^{2, \infty}$ in t, y . Moreover, we have the bounds

$$c e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \leq \|U_\varepsilon(t, \cdot)\|_{W_\alpha^{2, \infty}} \leq C e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}, \quad (4.1)$$

for positive constants c, C and σ_0 that do not depend on ε .

Inserting the expression for $u_\varepsilon, v_\varepsilon$ into the linearized Prandtl equation (1.6), we obtain

$$\begin{cases} \partial_t u_\varepsilon + u_s \partial_x u_\varepsilon + v_\varepsilon \partial_y u_s - \partial_y^2 u_\varepsilon = r_\varepsilon, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ \partial_x u_\varepsilon + \partial_y v_\varepsilon = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u, v)|_{y=0} = (0, 0). \end{cases} \quad (4.2)$$

The remainder term r_ε reads $r_\varepsilon = e^{i\varepsilon^{-1}x} R_\varepsilon(t, y)$, with

$$\begin{aligned} R_\varepsilon(t, y) &= e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \left(-\varepsilon^{-1} \left(u_s(t, y) - u_s(t, a(t)) - \partial_y^2 u_s(t, a(t)) \frac{y^2}{2} \right) \partial_y v_\varepsilon^{sl}(t, y) \right. \\ &\quad \left. + \varepsilon^{-1} \left(\partial_y u_s(t, a(t)) - \partial_y^2 u_s(t, a(t)) y \right) v_\varepsilon^{sl}(t, y) \right. \\ &\quad \left. - i \varepsilon \partial_y^3 v_\varepsilon^{reg}(t, y) + i \partial_t \partial_y \left(v_\varepsilon^{reg}(t, y) + v_\varepsilon^{sl}(t, y) \right) + O(\varepsilon^\infty) \right). \end{aligned}$$

The $O(\varepsilon^\infty)$ gathers terms with derivatives of φ : as the shear layer profile V decreases exponentially, and the derivatives of $\varphi(\cdot - a)$ are supported away from a , their contribution is indeed exponentially small. Straightforwardly,

$$\|R_\varepsilon(t, \cdot)\|_{W_\alpha^{0, \infty}} \leq C e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}, \quad (4.3)$$

with the same σ_0 as in (4.1).

We are now in a position to prove part i) of Theorem 1. Let us assume *a contrario* that for all $\sigma > 0$, there exists $m \geq 0$, $\mu \in [0, 1/2)$ and $\delta > 0$ such that

$$\sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t, s)\|_{\mathcal{L}(H^m, H^{m-\mu})} < +\infty.$$

Let

$$T_\varepsilon(t, s) : W_\alpha^{0, \infty}(\mathbb{R}_+) \mapsto W_\alpha^{0, \infty}(\mathbb{R}_+)$$

the restriction of $T(t, s)$ to the tangential Fourier mode ε^{-1} . Namely, $T(t, s) \left(e^{i\varepsilon^{-1}x} U_0 \right) = e^{i\varepsilon^{-1}x} T_\varepsilon(t, s) U_0$. Similarly, we denote $L_\varepsilon = e^{-i\varepsilon^{-1}x} L e^{i\varepsilon^{-1}x}$, where L is the linearized Prandtl operator around u_s . We have, for all $0 \leq s \leq t \leq \delta$,

$$\|T(t, s)\|_{\mathcal{L}(W_\alpha^{0, \infty})} \leq C \varepsilon^{-\mu} e^{\frac{\sigma(t-s)}{\sqrt{\varepsilon}}}.$$

Let $U = U(t, y)$ the solution of $\partial_t U + L_\varepsilon U = 0$, that coincides initially with the approximation U_ε . On one hand, we get

$$\|U(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C \varepsilon^{-\mu} e^{\frac{\sigma t}{\sqrt{\varepsilon}}} \|U(0, \cdot)\|_{W_\alpha^{0,\infty}} \leq C' \varepsilon^{-\mu} e^{\frac{\sigma t}{\sqrt{\varepsilon}}}. \quad (4.4)$$

On the other hand, the difference $\tilde{U} = U - U_\varepsilon$ satisfies, for all $t < \delta$,

$$\tilde{U}(t, \cdot) = \int_0^t T_\varepsilon(t, s) R_\varepsilon(s) ds.$$

Estimate (4.3) implies

$$\|\tilde{U}(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C \varepsilon^{-\mu} \int_0^t e^{\frac{\sigma(t-s)}{\sqrt{\varepsilon}}} e^{\frac{\sigma_0 s}{\sqrt{\varepsilon}}} ds \leq C' \varepsilon^{1/2-\mu} e^{\frac{\sigma_0 t}{\sqrt{|\varepsilon|}}},$$

as soon as $\sigma < \sigma_0$. Combining this with the estimate (4.1), we obtain the lower bound

$$\begin{aligned} \|U(t, \cdot)\|_{W_\alpha^{0,\infty}} &\geq \|U_\varepsilon(t, \cdot)\|_{W_\alpha^{0,\infty}} - \|\tilde{U}(t, \cdot)\|_{W_\alpha^{0,\infty}} \\ &\geq c e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} - C \varepsilon^{\mu-1/2} e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \end{aligned}$$

For ε small enough, we get

$$\|U(t, \cdot)\|_{W_\alpha^{0,\infty}} \geq c' e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}},$$

which contradicts the upper bound (4.4), as soon as $\sigma < \sigma_0$ and $t \gg \frac{\mu}{\sigma_0 - \sigma} |\ln(\varepsilon)| \sqrt{\varepsilon}$. This achieves the proof of part i).

4.2 Stronger ill-posedness for specific u_s

It remains to handle part ii) of Theorem 1. Roughly, we must find some u_s for which $e^{-\sigma\sqrt{|\partial_x|}(t-s)} T(t, s)$ fails to be bounded from H^m to $H^{m-\mu}$, $\mu \geq 0$ arbitrary. Using notations of the previous paragraph, the keypoint is to build, for any N , a growing solution $U_{\varepsilon, N}$ of

$$\partial_t U_{\varepsilon, N} + L_\varepsilon U_{\varepsilon, N} = R_{\varepsilon, N}, \quad \text{where } \|R_{\varepsilon, N}(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C_N (\varepsilon^N + t^{2N}) e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}.$$

Indeed, we can then take $N + 1/2 > \mu$, and conclude along the same lines as above.

So far, we have not managed to improve the approximation of the previous paragraph for general u_s . This explains the technical restriction $\mu \in [0, 1/2)$ of part i). In order to obtain a refined approximation, we consider some special profiles: we assume that $u_s(0, y) = U_s(y)$, for some exponentially decreasing U_s , satisfying in the neighborhood of $a > 0$:

$$U_s(y) = U_s''(a) \frac{(y-a)^2}{2}, \quad U_s''(a) < 0.$$

Notice that a is a non-degenerate critical point of U_s . For such profiles, the approximation of the previous paragraph reads

$$U_\varepsilon(t, y) = i e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_y \left(v_\varepsilon^{reg}(t, y) + v_\varepsilon^{sl}(t, y) \right), \quad \varepsilon = \frac{1}{n}, \quad n \in \mathbb{N}_*.$$

Using that U_s is quadratic near $y = a$, one can improve this approximation through an expansion of the type

$$U_{\varepsilon,N}(t, y) = U_\varepsilon(t, y) + i e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_y \sum_{i=1}^N \varepsilon^i v_\varepsilon^{i,reg}(t, y),$$

with additional terms $v_\varepsilon^{i,reg}$. Let us briefly explain the construction of these extra terms. The error terms due to U_ε divide into three categories:

1. Shear layer terms involving derivatives of φ . As mentioned before, they are $O(\varepsilon^\infty)$, and require no correction.
2. Terms that come from the replacement of u_s by its Taylor expansion in the shear layer equation. They read

$$\begin{aligned} R_{\varepsilon,1} &:= -\varepsilon^{-1} e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \left(u_s(t, y) - u_s(t, a(t)) - \partial_y^2 u_s(t, a(t)) \frac{y^2}{2} \right) \partial_y v_\varepsilon^{sl}(t, y), \\ R_{\varepsilon,2} &:= \varepsilon^{-1} e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \left(\partial_y u_s(t, a(t)) - \partial_y^2 u_s(t, a(t)) y \right) v_\varepsilon^{sl}(t, y). \end{aligned}$$

We write

$$\begin{aligned} |R_{\varepsilon,1}| &= \varepsilon^{-1} e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \left| \int_{a(t)}^y \frac{(z - a(t))^2}{2} \partial_y^3 u_s(t, z) dz \right| \left| \partial_y v_\varepsilon^{sl}(t, y) \right| \\ &\leq \varepsilon^{-1} e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \int_{a(t)}^y \frac{(z - a(t))^2}{2} \sum_{k=0}^{2N-1} \frac{t^k}{k!} \left| \partial_t^k \partial_y^3 u_s(0, z) \right| dz \left| \partial_y v_\varepsilon^{sl}(t, y) \right| + O(t^{2N}) e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \\ &\leq \varepsilon^{-1} e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}} \int_{a(t)}^y \frac{(z - a(t))^2}{2} \sum_{k=0}^{2N-1} \frac{t^k}{k!} \left| \partial_y^{3+2k} U_s(z) \right| dz \left| \partial_y v_\varepsilon^{sl}(t, y) \right| + O(t^{2N}) e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}. \end{aligned}$$

The second inequality stems from a Taylor expansion of u_s with respect to t . As u_s satisfies the heat equation, each time derivative can be replaced by two space derivatives, so the third line. Because U_s is quadratic in a vicinity of $a(t)$ (for short times), and v_ε^{sl} and its derivatives decay exponentially fast, we end up with

$$|R_{\varepsilon,1}| \leq C (t^{2N} + \varepsilon^N) e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}.$$

A similar bound holds for $R_{\varepsilon,2}$. Hence, these remainders do not require correction.

3. Terms that come from the time derivative and the diffusion. We focus here on the time derivative, as the diffusion term is simpler and has smaller amplitude. This is

$$R_{\varepsilon,3} := i e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_t \partial_y \left(v_\varepsilon^{reg}(t, y) + v_\varepsilon^{sl}(t, y) \right).$$

Proceeding as for $R_{\varepsilon,2}$, that is with Taylor expansions in t , leads to

$$\left| e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_t \partial_y v_\varepsilon^{sl}(t, y) \right| \leq C (t^{2N} + \varepsilon^N) e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}.$$

As regards the regular part,

$$\begin{aligned} e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_t \partial_y v_\varepsilon^{reg}(t, y) &= e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} H(y - a(t)) \partial_t \partial_y u_s \\ &= e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} H(y - a(t)) F(t, y) + O\left(t^{2N} e^{\frac{\sigma_0 t}{\sqrt{\varepsilon}}}\right) \end{aligned}$$

where

$$F(t, y) := \sum_{k=0}^{2N-1} \frac{t^k}{k!} \partial_y^{3+2k} U_s(y)$$

comes again from a Taylor expansion in t . The nice thing about this $O(1)$ term is that it is identically zero in the vicinity of $y = a(t)$ (for short times). As a result, the Heaviside function $H(y - a(t))$ in front of it does not create any discontinuity, and no extra shear layer term is necessary. One takes care of this source term by the introduction of

$$v_\varepsilon^{1,reg} = H(y - a(t)) \left(u_s(t, y) - \omega(\varepsilon, t) \right) \int_{a(t)}^y \frac{F(t, z)}{(u_s(t, z) - \omega(\varepsilon, t))^2} dz,$$

so that

$$U_\varepsilon^1 := -i\varepsilon e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} \partial_y v_\varepsilon^{1,reg}$$

solves

$$\partial_t U_\varepsilon^1 + L_\varepsilon U_\varepsilon^1 = i e^{i\varepsilon^{-1} \int_0^t \omega(\varepsilon, s) ds} H(y - a(t)) F(t, y) + O(\varepsilon).$$

Proceeding recursively, we obtain an approximation as accurate as we want. This ends the proof of the theorem.

5 Numerical study

In this last section, we present numerical illustrations of the instability process.

5.1 Numerical test of (SC)

To check (SC) numerically, it is more convenient to reformulate it with an *Evans function*. We know from Step 2, section 3, that there are solutions $\mathcal{Y}_\pm(z)$ of (3.6) satisfying (3.7)-(3.8). Back to the ODE (3.1), this yields independent solutions $X_+(\tau, \cdot)$ and $X_-(\tau, \cdot)$ respectively growing and decaying as z goes to $+\infty$. Furthermore, the following asymptotics holds:

$$\begin{aligned} X_\pm(\tau, z) &\sim z^{\pm \frac{i\tau}{2\lambda} - \frac{7}{2}} \exp\left(\pm \frac{1}{2} \lambda z^2\right), \\ \partial_z X_\pm(\tau, z) &\sim \pm \lambda z^{\pm \frac{i\tau}{2\lambda} - \frac{3}{2}} \exp\left(\pm \frac{1}{2} \lambda z^2\right), \end{aligned}$$

with $\lambda = \frac{1-i}{\sqrt{2}}$. Thus, the functions

$$W_-(\tau, z) := \int_z^{+\infty} X_-(\tau, s) ds, \quad W_+(\tau, z) := \int_0^z X_+(\tau, s) ds, \quad W_0(\tau, z) = 1,$$

seen as functions of z , form a basis of solutions of (1.7). They are respectively decaying, growing and constant at $+\infty$. As (1.7) is preserved by the change of variable $z \mapsto -z$, the

functions $W_-(\tau, -z)$, $W_+(\tau, -z)$, W_0 form a basis as well. They are respectively decaying, growing and constant at $-\infty$. The existence of the heteroclinic orbit is the same as the existence of some constants A and B such that

$$1 = AW_-(-z, \tau) + BW_-(z, \tau)$$

for all z , or equivalently

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \text{Vect} \left(\begin{pmatrix} W_-(\tau, 0) \\ \partial_z W_-(\tau, 0) \\ \partial_z^2 W_-(\tau, 0) \end{pmatrix}, \begin{pmatrix} W_-(\tau, 0) \\ -\partial_z W_-(\tau, 0) \\ \partial_z^2 W_-(\tau, 0) \end{pmatrix} \right)$$

This last condition is easily seen to be equivalent to

$$W_-(\tau, 0) \neq 0, \quad \text{and} \quad \partial_z^2 W_-(\tau, 0) = 0.$$

Hence, we must find τ with $\text{Im} \tau < 0$ such that

$$\int_0^{+\infty} X_-(\tau, s) ds \neq 0, \quad \text{and} \quad \partial_z X_-(\tau, 0) = 0.$$

Moreover, we know from Step 3, section 3 that for $\text{Im} \tau < 0$, the constraint $\int_0^{+\infty} X \neq 0$ is satisfied. Finally, the condition (SC) comes down to:

$$\partial_z X_-(\tau, 0) = 0, \quad \text{for some } \tau \text{ with } \text{Im} \tau < 0.$$

To check this, and get a value for τ , one can use a shooting method. For any τ and any $z_0 \gg 1$, one can start from the approximation

$$X_-(\tau, z_0) \approx z_0^{\frac{-i\tau}{2\lambda} - \frac{7}{2}} \exp(-\frac{1}{2}\lambda z_0^2), \quad \partial_z X_-(\tau, z_0) \approx -\lambda z_0^{\frac{-i\tau}{2\lambda} - \frac{3}{2}} \exp(-\frac{1}{2}\lambda z_0^2),$$

and integrate backwards (3.1) using a Runge-Kutta scheme. This gives access to the function $\partial_z X_-(\tau, z)$, for any τ and any $z \leq z_0$. Then, a Newton-Raphson procedure allows to find a zero in $\{\text{Im} \tau < 0\}$ for the function $\tau \mapsto \partial_z X_-(\tau, 0)$. Using such a procedure, we have found

$$\tau \approx -0.706 - 0.706 i.$$

Note that this value is proportional to $1 + i$, as expected from the analysis.

5.2 Simulation of the instability mechanism

To observe the instability mechanism described in section 2, we have performed direct simulations of system (2.1). More precisely, we have considered the velocity

$$u_s(t, y) = U_s(y) := 2y \exp(-y^2)$$

(already studied in [8] in the inviscid case), and solutions of the type

$$u_\varepsilon(t, x, y) = i e^{i\varepsilon^{-1}x} \partial_y V_\varepsilon(\varepsilon^{-1}t, y), \quad v_\varepsilon(t, y) = \varepsilon^{-1} V_\varepsilon(\varepsilon^{-1}t, y).$$

The profiles $V_\varepsilon = V_\varepsilon(\theta, y)$ satisfy the singular perturbation problem

$$(\partial_\theta + iU_s) \partial_y V_\varepsilon - iU'_s V_\varepsilon - \varepsilon e^{i\varepsilon^{-1}x} \partial_y^3 V_\varepsilon = 0$$

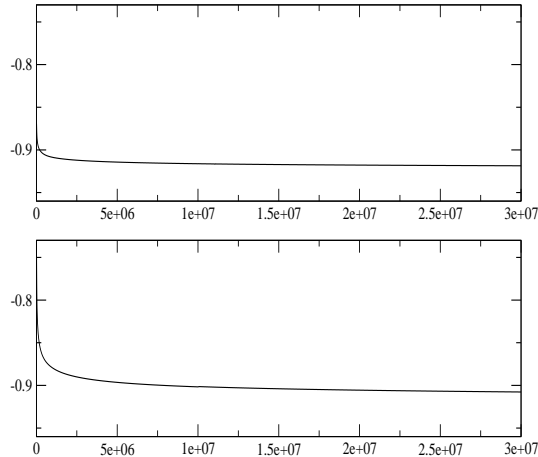


Figure 1: The correction to the inviscid eigenvalue. Plot of $\frac{1}{\sqrt{\varepsilon}} (\omega^{num}(\varepsilon) + U_s(a))$, seen as a function of the tangential frequency $k = \varepsilon^{-1}$. Top: real part. Bottom: imaginary part. As expected from the theory, both approximately converge to -0.9 , as k goes to infinity.

on $V_\varepsilon(\theta, y)$. One more differentiation gives the parabolic like equation

$$(\partial_\theta + iU_s) \partial_y^2 V_\varepsilon - i U_s'' V_\varepsilon - \varepsilon \partial_y^4 V_\varepsilon = 0,$$

fulfilled with the boundary conditions

$$V_\varepsilon|_{y=0} = \partial_y V_\varepsilon|_{y=0} = \partial_y^3 V_\varepsilon|_{y=0} = 0.$$

We have discretized this equation in space using finite differences on a stretched grid, and in time through a Crank-Nicholson scheme. Starting from initial random data (*i.e.* with random values at each gridpoint), we have computed its time evolution for values of $k = \varepsilon^{-1}$ ranging from 1 to $3 \cdot 10^7$. For sufficiently large times, one observes that the numerical solution V_ε^{num} behaves like

$$V_\varepsilon^{num}(\theta, y) \approx e^{i\omega^{num}(\varepsilon)\theta} v_\varepsilon^{num}(y)$$

in the sense that

$$\omega^{num}(\varepsilon) := \frac{V_\varepsilon^{num}(\theta + \Delta\theta, y) - V_\varepsilon^{num}(\theta, y)}{\Delta\theta V_\varepsilon^{num}(t, y)}$$

gets independent of θ and y . Computations show a relation of the type

$$\omega^{num}(\varepsilon) \sim -U_s(a) + \sqrt{\varepsilon}(-0.92 - 0.91i),$$

see figure 1. Here $a = \frac{1}{\sqrt{2}}$ is as usual the critical point of U_s . This relation is in very good agreement with the theoretical prediction.

$$\omega^{th}(\varepsilon) := -U_s(a) + \varepsilon^{1/2} \frac{|U_s''(a)|^{1/2} \tau}{\sqrt{2}} \approx -U_s(a) + \sqrt{\varepsilon}(-0.92 - 0.92i),$$

if we take for τ the value $-0.706 - 0.706i$ found in the previous subsection. Moreover, with this value of τ , one can compute directly the solution V of the shear layer equation. After proper

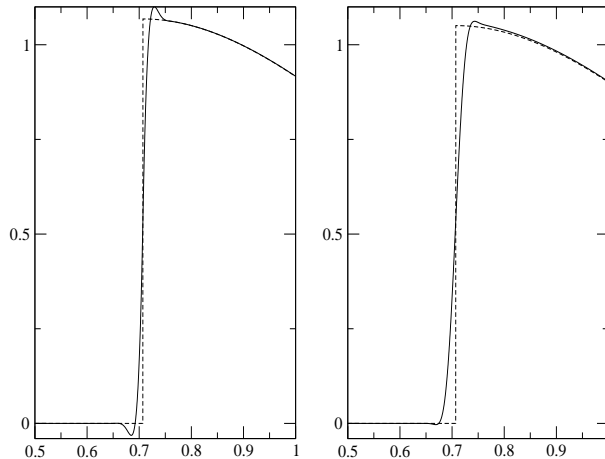


Figure 2: Plots of v_{out}^{th} (dashed line) and v_{out}^{num} (full line), seen as functions of y , at $\varepsilon = 10^{-7}$. Left and right figures correspond respectively to the real and imaginary parts. They match, as expected, outside the shear layer.

rescaling, this allows for comparison between the “numerical” and “theoretical” eigenmodes. More precisely, using the notations of (2.5), one can compare the functions

$$v_{out}^{th}(y) := \frac{1}{\sqrt{\varepsilon}} \left(\frac{v_{\varepsilon}^{reg}(y)}{v_{\varepsilon}^{reg}(\infty)} - \frac{v_a(y)}{v_a(\infty)} \right) \quad \text{and} \quad v_{out}^{num}(y) = \frac{1}{\sqrt{\varepsilon}} \left(\frac{v_{\varepsilon}^{num}(y)}{v_{\varepsilon}^{num}(\infty)} - \frac{v_a(y)}{v_a(\infty)} \right)$$

which should both describe the correction to the inviscid eigenmode *outside the shear layer*. As regards the shear layer, one can compare

$$v_{in}^{th}(z) := \frac{1}{\sqrt{\varepsilon}} \frac{v_{\varepsilon}^{sl}(\varepsilon^{1/4}(z+a))}{v_{\varepsilon}^{reg}(\infty)} \quad \text{and} \quad v_{in}^{num}(z) := \frac{1}{\sqrt{\varepsilon}} \frac{v_{\varepsilon}^{num}(\varepsilon^{1/4}(z+a))}{v_{\varepsilon}^{num}(+\infty)} - v_{out}^{num}(\varepsilon^{1/4}(z+a)).$$

Illustrations of these comparisons are given in figures 2 and 3. Again, we obtain an excellent agreement. This confirms that the instability mechanism we have described is indeed effective, and moreover dominates the linear dynamics (1.6).

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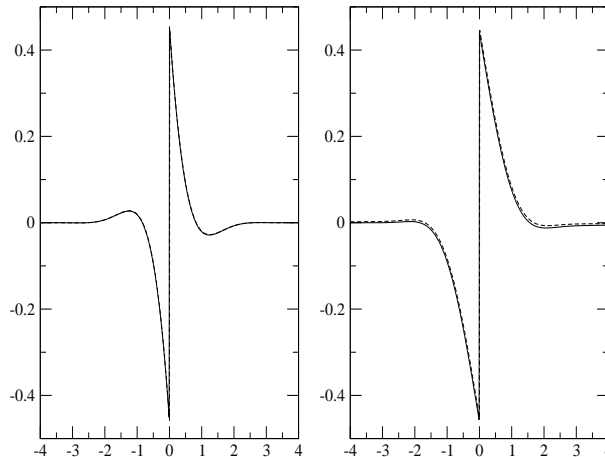


Figure 3: Plots of the shear layer corrections v_{in}^{th} (dashed line) and v_{in}^{num} (full line), seen as functions of z , at $\varepsilon = 10^{-7}$. Left and right figures correspond respectively to the real and imaginary parts.

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Appendix : Well-posedness in the analytic setting

We start from a simple estimate on the heat equation:

For $U_0 \in W_\alpha^{0,\infty}(\mathbb{R}_+)$, $F \in L^1(0, T; W_\alpha^{0,\infty}(\mathbb{R}_+))$, the solution U of

$$\partial_t U - \partial_y^2 U = F \quad \text{on } \mathbb{R} \times \mathbb{R}_+, \quad U|_{t=0} = U_0, \quad U|_{y=0} = 0,$$

satisfies

$$\|U\|_{L^\infty(W_\alpha^{0,\infty})} \leq C \left(\|U_0\|_{W_\alpha^{0,\infty}} + \|F\|_{L^1(W_\alpha^{0,\infty})} \right). \quad (5.1)$$

This estimate follows directly from the representation formula

$$U(t, y) = \int_{\mathbb{R}_+} S(t, y, z) U_0(z) dz + \int_0^t \int_{\mathbb{R}_+} S(t-s, y, z) F(s, z) dz ds$$

where the heat kernel in the half plane $S(t, y, z)$ reads

$$S(t, y, z) := G(t, y-z) - G(t, y+z), \quad G(t, y) := \frac{1}{\sqrt{4\pi t}} \exp(-y^2/4t).$$

Details are left to the reader. This estimate allows to prove proposition 1. Indeed, by decomposing

$$u(t, x, y) = \sum_{k \in \mathbb{Z}} e^{ikx} U^k(t, y),$$

the well-posedness is an easy consequence of the *a priori* estimate

$$\|U^k(t, \cdot)\|_{W^{0,\alpha}} \leq C e^{\rho kt} \|U^k(0, \cdot)\|_{W^{0,\alpha}}$$

for some ρ . Now, the equation satisfied by U^k is

$$\partial_t U^k - \partial_y^2 U^k = ik \left(U'_s \int_0^y U^k(t, z) dz - U_s U^k \right).$$

Using (5.1), we get

$$\|U^k(t, \cdot)\|_{W_\alpha^{0,\infty}} \leq C \|U^k(0, \cdot)\|_{W_\alpha^{0,\infty}} + C_s k \int_0^t \|U^k(s, \cdot)\|_{W_\alpha^{0,\infty}} ds$$

where C_s depends on u_s . We conclude by the Gronwall lemma.