OSCILLATING SOLUTIONS OF INCOMPRESSIBLE MHD AND DYNAMO EFFECT

DAVID GÉRARD-VARET *

Abstract. We are interested in the stability properties of some solutions of viscous incompressible MHD equations. These solutions are highly oscillating, with frequency involving a small parameter ε. They arise in the study of small-scale dynamo mechanisms. We prove both a nonlinear stability and instability results, depending on the time scale under consideration.

Key words. Magnetohydrodynamics, dynamo theory, oscillations.

AMS subject classifications. 35Q30, 35Q35

1. Introduction. This paper deals with some oscillatory solutions of the equations of magnetohydrodynamics (MHD). It is motivated by the study of small-scale dynamo mechanisms. Before we state precisely our main results, let us first specify the general framework.

The incompressible MHD equations read in a dimensionless form:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \dfrac{1}{\mathrm{Re}} \Delta u = \mathrm{curl}\, b \times b + f, \\ \partial_t b - \mathrm{curl}\,(u \times b) - \dfrac{1}{\mathrm{Rm}} \Delta b = 0, \\ \mathrm{div}\ u = \mathrm{div}\ b = 0. \end{cases} \tag{1.1}$$

They describe the evolution of an incompressible and electrically conducting fluid. They are derived from the incompressible Navier-Stokes equations, the Maxwell equations, and the Ohm's law in a conducting medium (see [15]). Functions

$$u = u(t, \mathrm{x}) \in \mathbb{R}^3, \quad b = b(t, \mathrm{x}) \in \mathbb{R}^3, \quad f = f(t, \mathrm{x}) \in \mathbb{R}^3$$

model respectively the fluid velocity, the magnetic field, and an additional forcing term, for instance due to convection. The space and time variables are $t \in \mathbb{R}^+$, $\mathrm{x} = (x, y, z) \in \mathbb{R}^3$. We denote

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad \nabla = (\partial_x, \partial_y, \partial_z)^t,$$

and for any $v = (v_1(\mathrm{x}), v_2(\mathrm{x}), v_3(\mathrm{x}))^t \in \mathbb{R}^3$,

$$\mathrm{div}\ v = \partial_x v_1 + \partial_y v_2 + \partial_z v_3, \quad \mathrm{curl}\, v = (\partial_y v_3 - \partial_z v_2, \partial_z v_1 - \partial_x v_3, \partial_x v_2 - \partial_y v_1)^t.$$

Constants Re and Rm are called hydrodynamic and magnetic Reynolds numbers. To lighten notations, we will assume in the sequel that Re = Rm = 1. Note that the divergence-free condition on b is preserved by equation (1.1)b. As soon as it is satisfied initially, it is satisfied for all positive times.

Roughly speaking, *dynamo theory* deals with the stability of solutions

$$(u, b) = (u(t, \mathrm{x}), 0)$$

*D.M.A, U.M.R. 8553, E.N.S., 45 Rue d'Ulm, 75230 Paris Cedex 05 (David.Gerard-Varet@ens.fr)

of system (1.1). More precisely, it studies the generation of magnetic field from the fluid flow u. The basic idea is that the "self-excited" term $\mathrm{curl}\,(u \times b)$ may amplify the magnetic field in the manner of an instability. As long as the fluid motion is strong enough, this transfer from kinetic to magnetic energy may thus prevent the decay of the magnetic field, despite the dissipation term $-\Delta b$.

It is widely accepted that dynamo action takes place in the Earth, in the Sun, and in many other planets and stars. Therefore, the understanding of dynamo mechanisms is a major physical issue. It has been the matter of a huge literature: we refer to the recent review papers by Gilbert [8] and Fearn [5] for a good introduction and appropriate lists of references. Note that most of these references are limited to *kinematic dynamos*: the Laplace force is neglected, and only the induction equation (1.1b) is considered, at imposed velocity u.

Among the mechanisms that have been identified, one of the most famous is the so-called *alpha effect*. It is based on a *scale separation*: the velocity and magnetic fields are assumed to vary on (turbulent) time and length scales τ and l, much smaller than the typical macro scales T and L. Introducing the ratios $\lambda = \tau/T \ll 1$ and $\beta = l/L \ll 1$, one can write this with little formalism:

$$\begin{aligned} u &\approx u_* \left(t, \mathrm{x}, \lambda^{-1} t,\, \beta^{-1}\mathrm{x}\right) \;+\; \bar{u}\,(t, \mathrm{x}) \,, \\ b &\approx b_* \left(t, \mathrm{x}, \lambda^{-1} t,\, \beta^{-1}\mathrm{x}\right) \;+\; \bar{b}\,(t, \mathrm{x}) \,, \end{aligned} \tag{1.2}$$

where u_* (resp. b_*) is the fluctuating part of the field, and $\bar{u}$ (resp. $\bar{b}$) is its mean part. The basic idea is that the "average" of the fluctuating term $\mathrm{curl}\,(u_* \times b_*)$ can have a destabilizing effect on the mean field $\bar{b}$, generating a dynamo.

This idea has been first introduced by Parker [13] in 1955, and in a geophysical context by Braginsky [2]. It has been generalized by Steenbeck, Krause, and Radler [16]. Let us also mention the important works [14] and [3], on periodic dynamos. Note that the alpha effect has since been confirmed experimentally [17].

The present paper is a small step towards the mathematical study of this mechanism. Namely, we will investigate the stability properties of solutions $(u, 0)$ of (1.1) given by

$$(u, 0) = (\varepsilon^{-1} u^\varepsilon, 0), \quad u^\varepsilon(t, \mathrm{x}) = U\Big(\varepsilon^{-4} t, \varepsilon^{-2}\mathrm{x}\Big), \tag{1.3}$$

where $U = U(\tau, \theta)$ satisfies

$$U \in H^\infty \left(\mathbb{T} \times \mathbb{T}^3\right)^3, \quad \int_{\mathbb{T}\times\mathbb{T}^3} U = 0, \quad \mathrm{div}_\theta\, U = 0. \tag{1.4}$$

Remark 1.1. *The set $\mathcal{P}$ of profiles U satisfying* (1.4) *is a Frechet space, where the topology is induced by the family of norms*

$$\|U\|_m^2 = \sum_{(\omega,\xi)\in\mathbb{Z}^4} \left(|\omega|^2 + |\xi|^2\right)^m |\hat{U}(\omega, \xi)|^2, \quad m \geq 0,$$

where $\hat{U}$ is the Fourier transform with respect to (τ, θ). We denote by $d_{\mathcal{P}}$ a metric defining this topology.

REMARK 1.2. *We assume that $\varepsilon \ll 1$ and that $\int U = 0$, which means we consider fast oscillations with zero mean flow. This is reminiscent of the (somehow crude) modeling of turbulence that we have in mind.*

REMARK 1.3. *The amplitude, time and length scales in* (1.3) *are classical (see [8]). To fix ideas, one can say using notations of* (1.2) *that they correspond to the case*

$$|u_*| \gg |\bar{u}|, \quad |b_*| \ll |\bar{b}|.$$

Indeed, the oscillatory part $u_* = \varepsilon^{-1} u^\varepsilon$ of the velocity field is $O(\varepsilon^{-1})$, bigger than the $O(1)$ potential mean part $\bar{u}$. On the contrary, our choices $O(\varepsilon^4)$ and $O(\varepsilon^2)$ for time and length scales of u^ε ensure small amplitude for the oscillatory part of b. This can be seen formally from equation (1.1b), as the oscillation $b_* = b_*(t, \mathrm{x}, \tau = \varepsilon^{-4} t, \theta = \varepsilon^{-2} \mathrm{x})$ satisfies:

$$\varepsilon^{-4} \left(\partial_\tau b_* - \Delta_\theta b_* \right) = o(1)$$

We emphasize that other choices for solutions u^ε are possible. In particular, it would be interesting to study oscillating fields with larger wavelength, so as to emphasize the role of the hyperbolic part of (1.1).

We will show that solutions u^ε given by (1.3) *are stable on times $t = O(1)$, and "generically" unstable on times $t \sim |\ln(\varepsilon)|$.* Substituting $u = \varepsilon^{-1} u^\varepsilon + v$ into (1.1), we will rather work with the system:

$$\begin{cases} \partial_t v + \varepsilon^{-1} u^\varepsilon \cdot \nabla v + \varepsilon^{-1} v \cdot \nabla u^\varepsilon + v \cdot \nabla v + \nabla p - \Delta v = \operatorname{curl} b \times b, \\ \partial_t b - \varepsilon^{-1} \operatorname{curl} \left(u^\varepsilon \times b \right) - \operatorname{curl} \left(v \times b \right) - \Delta b = 0, \\ \operatorname{div} \, v = \operatorname{div} \, b = 0, \end{cases} \tag{1.5}$$

Note that for all $\varepsilon > 0$ and all divergence-free fields $v_0^\varepsilon, b_0^\varepsilon \in L^2 \left(\mathbb{R}^3 \right)^3$, system (1.5) has global weak solutions

$$v, b \in L^\infty_{loc} \left(\mathbb{R}^+; \, L^2 \left(\mathbb{R}^3 \right) \right)^3 \cap L^2_{loc} \left(\mathbb{R}^+; \, \dot{H}^1 \left(\mathbb{R}^3 \right) \right)^3,$$

with initial data $v_0^\varepsilon, b_0^\varepsilon$. Indeed, u^ε and its derivatives are bounded functions, so that classical Leray type existence theorem for MHD equations extends easily to (1.5).

We will first prove the following stability result

THEOREM 1.4. **(Nonlinear stability result)**

Let $U \in \mathcal{P}$, and $\{u^\varepsilon\}_{\varepsilon > 0}$ satisfying (1.3). *Let $m \in \mathbb{N}$, and v_0, b_0 in $H^\infty \left(\mathbb{R}^3 \right)^3$, divergence-free.*

For all $T \geq 0$, there exists $\delta > 0$, $\varepsilon_0 > 0$ such that: if

$$m \geq 1 \quad \text{or} \quad \|(v_0, b_0)\|_{H^{1/2}} \leq \delta,$$

the Cauchy problem (1.5) *with initial data $\varepsilon^m v_0, \varepsilon^m b_0$ has a unique solution*

$$v^\varepsilon, b^\varepsilon \in C^0 \left([0, T]; \, H^\infty \left(\mathbb{R}^3 \right) \right)^3$$

for all $\varepsilon < \varepsilon_0$. Moreover, it satisfies for a positive constant C, large s and small enough ε:

$$\sup_{0\le t\le T} \|(v^\varepsilon, b^\varepsilon)(t,\cdot)\|_{L^2} \;\le\; C\,\varepsilon^m\,\|(v_0,b_0)\|_{H^s},$$
$$\sup_{0\le t\le T} \|(v^\varepsilon, b^\varepsilon)(t,\cdot)\|_{L^\infty} \;\le\; C\,\varepsilon^m\,\|(v_0,b_0)\|_{H^s}.$$

We will then prove the following instability result:

THEOREM 1.5. **(Nonlinear instability result)**

There exists a dense and open subset Ω of $\mathcal{P}$ such that:

- *for all $U \in \Omega$, and $\{u^\varepsilon\}_{\varepsilon>0}$ satisfying* (1.3),
- *for all $m \in \mathbb{N}$,*

one can find $\delta > 0$, times $t(\varepsilon) = O(|\ln(\varepsilon)|)$, and families of solutions $\{(v^\varepsilon, b^\varepsilon)^t\}_{\varepsilon>0}$ of (1.5) *with*

$$v^\varepsilon, b^\varepsilon \in C^0\left(\mathbb{R}^+;\, H^\infty\left(\mathbb{R}^3\right)\right)^3,$$

$$\|\,\partial_{\mathrm{x}}^\alpha (v^\varepsilon, b^\varepsilon)|_{t=0}\,\|_{L^2} \le C_\alpha\,\varepsilon^{m-2|\alpha|+1}, \quad \forall \alpha \in \mathbb{N}^3,$$

and

$$\|b^\varepsilon|_{t=t(\varepsilon)}\|_{L^2} \ge \delta.$$

REMARK 1.6. *Note that the lower bound in theorem 1.5 applies to b^ε, which is exactly the mathematical expression of an alpha effect: small-scale velocity u^ε generates destabilization of $b = 0$.*

REMARK 1.7. *Theorem* (1.5) *extends and justifies linear computations carried in [8]. It is also reminiscent of classical linear computations of G.O Roberts [14] on periodic dynamos. Roberts studies in [14] the equation*

$$\partial_t b + curl\,(u^\varepsilon \times b) - \nu\varepsilon\Delta b = 0, \quad u^\varepsilon = U\left(\varepsilon^{-1}t, \varepsilon^{-1}\mathrm{x}\right),$$

for U periodic with zero mean. He shows an instability result for ν large enough. However, his analysis, which relies heavily on perturbation theory, does not adapt to our nonlinear framework. Henceforth, we use a strongly different approach, based on energy estimates. The paper is structured as follows. In section 2, we introduce an auxiliary singular system. This system involves additional variables, which take into account the dependence of $(v^\varepsilon, b^\varepsilon)$ on $\varepsilon^{-4}t$, $\varepsilon^{-2}\mathrm{x}$. We construct approximate solutions of this system (section 2.1), and perform a priori estimates (section 2.2). The proof of theorem 1.4 follows (section 2.3). In section 3, we focus on the instability mechanism. We show that the approximate solutions of section 2 have generically exponential growth (section 3.1), and give precise estimates on this growth (section 3.2). We end with the proof of theorem 1.5 (section 3.3).

2. Singular system. From the structure of the small-scale flow u^ε, we expect solutions $v^\varepsilon, b^\varepsilon$ of (1.5) to exhibit rapid oscillations, involving $\varepsilon^{-4}t$ and $\varepsilon^{-2}\mathrm{x}$. In particular, we expect the derivatives of $v^\varepsilon, b^\varepsilon$ to behave badly. Hence, we do not expect good H^s energy estimates on system (1.5). To override this difficulty, we will follow ideas of nonlinear geometric optics (*cf* [10]): we will work directly in the class of solutions of the following type:

$$(v^\varepsilon, b^\varepsilon)^t(t,\mathrm{x}) = V^\varepsilon\left(t, \mathrm{x}, \varepsilon^{-4}t, \varepsilon^{-2}\mathrm{x}\right), \tag{2.1}$$

where $V^\varepsilon = V^\varepsilon(t,\mathrm{x},\tau,\theta)$ is periodic in τ and θ. We will get Sobolev bounds on V^ε, which will allow to control v^ε, b^ε in L^2 and L^∞ (*cf* theorem 1.5).

We introduce the following singular system, of unknown $V = (w,\beta)^t$:

$$\begin{cases} \partial_t V + \varepsilon^{-4}\partial_\tau V + \varepsilon^{-1}(B_\mathrm{x} + \varepsilon^{-2}B_\theta)(\tilde{U}, V) + \dfrac{1}{2}(B_\mathrm{x} + \varepsilon^{-2}B_\theta)(V,V) \\ -\left(\nabla_\mathrm{x} + \varepsilon^{-2}\nabla_\theta\right)^2 V = \left(\left(\nabla_\mathrm{x} + \varepsilon^{-2}\nabla_\theta\right)p,\, 0\right)^t, \\ \mathrm{Div}_\mathrm{x} V + \varepsilon^{-2}\mathrm{Div}_\theta V = 0, \end{cases} \tag{2.2}$$

with $\tilde{U} := (U,0)^t$, $U = U(\tau,\theta) \in \mathcal{P}$, where for all $V = (w,\beta)$, $\tilde{V} = (\tilde{w},\tilde{\beta})$,

$$B_\mathrm{x}(V,\tilde{V}) := \begin{pmatrix} \mathrm{div}_\mathrm{x}\left(w\otimes\tilde{w} - \beta\otimes\tilde{\beta}\right) + \mathrm{div}_\mathrm{x}\left(\tilde{w}\otimes w - \tilde{\beta}\otimes\beta\right) \\ -\mathrm{curl}_\mathrm{x}\left(w\times\tilde{\beta}\right) - \mathrm{curl}_\mathrm{x}\left(\tilde{w}\times\beta\right) \end{pmatrix},$$

$$B_\theta(V,\tilde{V}) := \begin{pmatrix} \mathrm{div}_\theta\left(w\otimes\tilde{w} - \beta\otimes\tilde{\beta}\right) + \mathrm{div}_\theta\left(\tilde{w}\otimes w - \tilde{\beta}\otimes\beta\right) \\ -\mathrm{curl}_\theta\left(w\times\tilde{\beta}\right) - \mathrm{curl}_\theta\left(\tilde{w}\times\beta\right) \end{pmatrix},$$

and where

$$\mathrm{Div}_\mathrm{x} = \begin{pmatrix} \mathrm{div}_\mathrm{x} & \\ & \mathrm{div}_\mathrm{x} \end{pmatrix}, \quad \mathrm{Div}_\theta = \begin{pmatrix} \mathrm{div}_\theta & \\ & \mathrm{div}_\theta \end{pmatrix}.$$

Remark that the quadratic terms in (1.5a) satisfy:

$$u^\varepsilon\cdot\nabla v + v\cdot\nabla u^\varepsilon = \mathrm{div}\,(u^\varepsilon\otimes v) + \mathrm{div}\,(v\otimes u^\varepsilon),$$

$$\mathrm{curl}\, b\times b = b\cdot\nabla b + \frac{1}{2}\nabla|b|^2 = \mathrm{div}\,(b\otimes b) + \frac{1}{2}\nabla|b|^2,$$

using that v, b are divergence free. We thus see that any regular solution V^ε of (2.2) provides a solution $v^\varepsilon, b^\varepsilon$ of (1.5), through identity (2.1).

2.1. Approximate solutions. *Up to the end of the section, we fix time $T > 0$, and $m \in \mathbb{N}$.* For any

$$f = f(t,\mathrm{x},\tau,\theta) = \sum_{\omega,k} f_{\omega,k}(t,\mathrm{x})\, e^{i(\omega\tau + k\cdot\theta)},$$

we denote $\overline{f} = f_{0,0}$, $f_* = f - \overline{f}$.

We first construct approximate solutions of (2.2) of the following type:

$$\begin{aligned} V^\varepsilon(t,\mathrm{x},\tau,\theta) &\approx \varepsilon^m \sum \varepsilon^i V^i(t,\mathrm{x},\tau,\theta), \\ p^\varepsilon(t,\mathrm{x},\tau,\theta) &\approx \varepsilon^{m-1} \sum \varepsilon^i p^i(t,\mathrm{x},\tau,\theta), \end{aligned} \tag{2.3}$$

where for all $i \geq 0$,

$$\left(V^i, p^i\right)^t = (w^i, \beta^i, p^i)^t \in C^\infty\left([0,T];\, H^\infty\left(\mathbb{R}^3 \times \mathbb{T} \times \mathbb{T}^3\right)\right)^7, \quad \int \overline{p}^i(t,\cdot) = 0.$$

We plug approximation (2.3) in the system (2.2). We identify terms of order ε^{m+i-4} in (2.2a) and of order ε^{m+i-2} in (2.2b). It yields, for all $i \geq 0$:

$$\begin{cases} (\partial_\tau - \Delta_\theta)V^i = F^i, \\ \mathrm{Div}_\theta V^i = -\mathrm{Div}_\mathrm{x} V^{i-2}, \end{cases} \tag{S_i}$$

where $V^j := 0$, $p^j := 0$ for $j < 0$, and

$$\begin{aligned} F^i = &-(\partial_t - \Delta_\mathrm{x})V^{i-4} - B_\mathrm{x}\left(\tilde{U}, V^{i-3}\right) + \left(\mathrm{div}_\mathrm{x}\nabla_\theta + \mathrm{div}_\theta \nabla_\mathrm{x}\right)V^{i-2} \\ &- \sum_{j+J=i-m-4} B_\mathrm{x}\left(V^j, V^J\right) - \sum_{j+J=i-m-2} B_\theta\left(V^j, V^J\right) - B_\theta\left(\tilde{U}, V^{i-1}\right) \\ &+ \left(\nabla_\theta p^{i-1}, 0\right)^t + \left(\nabla_\mathrm{x} p^{i-3}, 0\right)^t. \end{aligned} \tag{2.4}$$

Note that (S_0) is equivalent to $V^0_* \equiv 0$. For $i \geq 0$, we take the oscillatory part of (S_{i+1}), and the average of (S_{i+4}). We get:

$$\begin{cases} (\partial_\tau - \Delta_\theta)V^{i+1}_* = \left(\nabla_\theta p^i_*, 0\right)^t - B_\theta\left(\tilde{U}, \overline{V}^i\right) + G^i_* \\ \mathrm{Div}_\theta V^{i+1}_* = -\mathrm{Div}_\mathrm{x} V^{i-1}_*, \\ (\partial_t - \Delta_\mathrm{x})\overline{V}^i = \left(\nabla_\mathrm{x}\overline{p}^{i+1}, 0\right)^t - \overline{B_\mathrm{x}\left(\tilde{U}, V^{i+1}_*\right)} - \sum_{j+J=i-m} \overline{B_\mathrm{x}\left(V^j, V^J\right)} \\ \mathrm{Div}_\mathrm{x}\overline{V}^i = 0. \end{cases} \tag{T_i}$$

where $G^i_* := F^{i+1}_* - \left(\nabla_\theta p^i_*, 0\right)^t + B_\theta\left(\tilde{U}, \overline{V}^i\right)$ depends only on V^0, $\ldots$, V^{i-1}, V^i_*, and $\nabla_\mathrm{x} p^{i-1}_*$.

Thus, introducing

$$X^i := \left(\overline{V}^i, V^{i+1}_*, p^i_*, \overline{p}^{i+1}\right),$$

(T_i) can be seen as a system of unknown X^i, with data depending on X^0, X^1, $\ldots$, X^{i-1}. We will show inductively on $i \geq 0$ the solvability of (T_i).

<u>Case $i = 0$</u>

Remind that $V^0_* \equiv 0$. As $\overline{p}^0$ does not appear in systems (T_i), we can also assume $\overline{p}^0 \equiv 0$. System (T_0) depends on the values of m.

- $m = 0$

The system (T_0) reads

$$\begin{cases} (\partial_\tau - \Delta_\theta)V_*^1 = \left(\nabla_\theta p_*^0, 0\right)^t - B_\theta\left(\tilde{U}, \overline{V}^0\right) \\ \mathrm{Div}_\theta V_*^1 = 0, \\ (\partial_t - \Delta_{\mathrm{x}})\overline{V}^0 = \left(\nabla_{\mathrm{x}}\overline{p}^1, 0\right)^t - \overline{B_{\mathrm{x}}\left(\tilde{U}, V_*^1\right)} - B_{\mathrm{x}}\left(\overline{V}^0, \overline{V}^0\right), \\ \mathrm{Div}_{\mathrm{x}}\overline{V}^0 = 0. \end{cases}$$

Applying Div_θ to (T_0a) and using (T_0b) leads to

$$(\Delta_\theta p_*^0, 0)^t = \mathrm{Div}_\theta B_\theta\big(\tilde{U}, \overline{V}^0\big).$$

As the second component of the right hand side is $\mathrm{div}_\theta\,\mathrm{curl}_\theta\big(U \times \overline{\beta}^0\big) \equiv 0$, such equation has a unique solution

$$(p_*^0, 0)^t = \Delta_\theta^{-1}\mathrm{Div}_\theta B_\theta\big(\tilde{U}, \overline{V}^0\big). \tag{2.5}$$

Here, Δ_θ^{-1} denotes the inverse of Δ_θ, in the set of L^2 periodic functions of (τ, θ) with zero average in θ. Thus, we are left with:

$$\begin{cases} (\partial_\tau - \Delta_\theta)V_*^1 = L_\theta \overline{V}^0, \\ (\partial_t - \Delta_{\mathrm{x}})\overline{V}^0 = \left(\nabla_{\mathrm{x}}\overline{p}^1, 0\right)^t - \overline{B_{\mathrm{x}}\left(\tilde{U}, V_*^1\right)} - B_{\mathrm{x}}\left(\overline{V}^0, \overline{V}^0\right), \\ \mathrm{Div}_{\mathrm{x}}\overline{V}^0 = 0. \end{cases}$$

where

$$L_\theta \overline{V}^0 := \nabla_\theta \Delta_\theta^{-1}\mathrm{Div}_\theta B_\theta\left(\tilde{U}, \overline{V}^0\right) - B_\theta\left(\tilde{U}, \overline{V}^0\right).$$

Note that

$$V_*^1 = (\partial_\tau - \Delta_\theta)^{-1} L_\theta \overline{V}^0, \tag{2.6}$$

where $(\partial_\tau - \Delta_\theta)^{-1}$ denotes the inverse of $\partial_\tau - \Delta_\theta$ in the set of L^2 periodic functions of (τ, θ) with zero average in θ. We end up with

$$\begin{cases} (\partial_t - \Delta_{\mathrm{x}})\overline{V}^0 = \left(\nabla_{\mathrm{x}}\overline{p}^1, 0\right)^t - \overline{B_{\mathrm{x}}\left(\tilde{U}, (\partial_\tau - \Delta_\theta)^{-1}L_\theta\overline{V}^0\right)} - B_{\mathrm{x}}\left(\overline{V}^0, \overline{V}^0\right), \\ \mathrm{Div}_{\mathrm{x}}\overline{V}^0 = 0. \end{cases} \tag{2.7}$$

This last system is of the type (1.1), up to the additional linear term

$$\mathcal{A}\overline{V}^0 := -\overline{B_{\mathrm{x}}\left(\tilde{U}, (\partial_\tau - \Delta_\theta)^{-1}L_\theta\overline{V}^0\right)}.$$

Classical existence results for smooth solutions of Navier-Stokes type equations extend without difficulty (see [7]). In particular, there exists $\delta = \delta(T)$ such that: for all

$$\overline{V}_0^0 \in H^\infty\left(\mathbb{R}^3\right)^6, \quad \mathrm{Div}_{\mathrm{x}}\overline{V}_0^0 = 0, \quad \|\overline{V}_0^0\|_{H^{1/2}} \le \delta,$$

system (2.7) has a unique solution

$$\left(\overline{V}^0, \overline{p}^1\right) \in C^\infty\left([0,T];\ H^\infty\left(\mathbb{R}^3\right)\right)^7, \quad \int \overline{p}^1(t,\cdot) = 0,$$

with $\overline{V}^0|_{t=0} = \overline{V}^0_0$. Together with (2.5) and (2.6), it provides a unique solution

$$X^0 = \left(\overline{V}^0, V^1_*, p^0_*, \overline{p}^1\right)$$

of system (T_0).

- $m \geq 1$

The situation is even simpler, as the quadratic term disappears. We are left with

$$\begin{cases} (\partial_t - \Delta_{\mathrm{x}})\overline{V}^0 = \left(\nabla_{\mathrm{x}}\overline{p}^1, 0\right)^t + \mathcal{A}\overline{V}^0, \\ \mathrm{Div}_{\mathrm{x}}\overline{V}^0 = 0. \end{cases} \tag{2.8}$$

which has regular solutions

$$\left(\overline{V}^0, p^1\right)^t \in C^0\left(\mathbb{R}^+;\ H^\infty\left(\mathbb{R}^3\right)\right)^7, \quad \int \overline{p}^1(t,\cdot) = 0.$$

for any initial data $\overline{V}^0_0$ in $H^\infty\left(\mathbb{R}^3\right)^6$, $\mathrm{Div}_{\mathrm{x}}\overline{V}^0 = 0$.

<u>Case $i \geq 1$</u>

The solvability of (T_i) is proved inductively. Let $i \geq 1$, and $X^0, \ldots, X^{i-1}$ solving $(T_0), \ldots, (T_{i-1})$ on the time interval $[0,T]$. Applying Div_θ to $(T_i a)$ and using $(T_i b)$ leads to

$$(\Delta_\theta p^i_*, 0)^t = \mathrm{Div}_\theta\left(B_\theta\left(\tilde{U}, \overline{V}^i\right) - G^i_*\right) - (\partial_\tau - \Delta_\theta)\,\mathrm{Div}_{\mathrm{x}} V^{i-1}_*. \tag{2.9}$$

To solve (T_i), we need to verify that the right hand side of (2.9) has zero second component. This compatibility condition ensures that the magnetic component of our approximation remains divergence-free. It is reminiscent to the fact that $\mathrm{div}\, b = 0$ is preserved by (1.5)b.

If we denote $G^i_* = \left(g^i_*, h^i_*\right)^t$, equation (2.9) has a unique solution if and only if

$$\mathrm{div}_\theta h^i_* = -(\partial_\tau - \Delta_\theta)\mathrm{div}_{\mathrm{x}}\beta^{i-1}_*,$$

i.e.

$$\mathrm{div}_\theta h^i_* = -\mathrm{div}_{\mathrm{x}} h^{i-2}_*. \tag{2.10}$$

The expression of h^i_* yields

$$\begin{aligned} \mathrm{div}_\theta h^i_* = \mathrm{div}_\theta\Bigg(& -(\partial_t - \Delta_{\mathrm{x}})\beta^{i-3} + \left(\mathrm{div}_{\mathrm{x}}\nabla_\theta + \mathrm{div}_\theta\nabla_{\mathrm{x}}\right)\beta^{i-1} \\ & + \mathrm{curl}_{\mathrm{x}}\left(U \times \beta^{i-2}\right) + \mathrm{curl}_{\mathrm{x}} \sum_{j+J=i-m-3} w^j \times \beta^J\Bigg) \\ = & -(\partial_t - \Delta_{\mathrm{x}})\mathrm{div}_\theta\beta^{i-3} + \left(\mathrm{div}_{\mathrm{x}}\nabla_\theta + \mathrm{div}_\theta\nabla_{\mathrm{x}}\right)\mathrm{div}_\theta\beta^{i-1} \\ & - \mathrm{div}_{\mathrm{x}}\mathrm{curl}_\theta\left(U \times \beta^{i-2}\right) - \mathrm{div}_{\mathrm{x}}\mathrm{curl}_\theta \sum_{j+J=i-m-3} w^j \times \beta^J. \end{aligned}$$

Using that $\mathrm{div}_\theta \beta^i = -\mathrm{div}_\mathrm{x} \beta^{i-2}$ for all i and $\mathrm{div}_\mathrm{x}\mathrm{curl}_\mathrm{x} \equiv 0$, we get

$$\begin{aligned}\mathrm{div}_\theta h_*^i = -\mathrm{div}_\mathrm{x}\Bigg(& -(\partial_t - \Delta_\mathrm{x})\beta^{i-5} + \left(\mathrm{div}_\mathrm{x}\nabla_\theta + \mathrm{div}_\theta \nabla_\mathrm{x}\right)\beta^{i-3} \\ & + \mathrm{curl}_\theta \left(U \times \beta^{i-2}\right) + \mathrm{curl}_\mathrm{x}\left(U \times \beta^{i-4}\right) \\ & + \mathrm{curl}_\theta \sum_{j+J=i-m-3} w^j \times \beta^J + \mathrm{curl}_\mathrm{x} \sum_{j+J=i-m-5} w^j \times \beta^J \Bigg),\end{aligned}$$

which is exactly (2.10). Hence,

$$\left(p_*^i, 0\right)^t = \Delta_\theta^{-1} \mathrm{Div}_\theta B_\theta \left(\tilde{U}, \overline{V}^i\right) + H_*^i, \tag{2.11}$$

where H_*^i depends only on $X^0, \ldots, X^{i-1}$. Solving $(T_i a)$ yields in turn

$$V_*^{i+1} = (\partial_\tau - \Delta)^{-1} L_\theta \overline{V}^i + I_*^i,$$

where I_*^i depends only on $X^0, \ldots, X^{i-1}$. We are left with equations of the following type:

$$\begin{cases} (\partial_t - \Delta_\mathrm{x})\overline{V}^i = \left(\nabla_\mathrm{x} \overline{p}^{i+1}, 0\right)^t + \mathcal{A}\overline{V}^i + \overline{J}^i \\ \qquad\qquad - \delta_{0m}\left(B_\mathrm{x}\left(\overline{V}^0, \overline{V}^i\right) + B_\mathrm{x}\left(\overline{V}^i, \overline{V}^0\right)\right), \\ \mathrm{Div}_\mathrm{x} \overline{V}^i = 0, \end{cases} \tag{2.12}$$

where δ_{0m} is the Kronecker symbol, and where

$$\overline{J}^i \in C^\infty\left([0,T];\, H^\infty\left(\mathbb{R}^3\right)\right)^6$$

depends on $X^0, \ldots, X^{i-1}$. This system is linear, and of parabolic type. It follows easily that, for all initial data $\overline{V}^i|_{t=0} = \overline{V}_0^i$ in $H^\infty\left(\mathbb{R}^3\right)^6$, $\mathrm{Div}_\mathrm{x}\overline{V}_0^i = 0$, such system has a unique solution

$$\left(\overline{V}^i, \overline{p}^i\right) \in C^\infty\left([0,T];\, H^\infty\left(\mathbb{R}^3\right)\right)^7, \quad \int \overline{p}^i(t,\cdot) = 0.$$

Back to system (T_i), this ends the induction.

2.2. A priori estimates. We now establish some stability estimates on systems of type (2.2), that will be used in sections 2.3 and 3.3. More precisely, let $\{T^\varepsilon\}_{\varepsilon>0}$ a family of times, and $\left\{V_{app}^\varepsilon\right\}_{\varepsilon>0}$, $\{F^\varepsilon\}_{\varepsilon>0}$ families of functions satisfying

$$\forall \varepsilon, \quad V_{app}^\varepsilon, F_{app}^\varepsilon \in C\left([0,T^\varepsilon];\, H^\infty\left(\mathbb{T}\times\mathbb{T}^3\times\mathbb{R}^3\right)\right)^6,$$

and such that

$$\sup_\varepsilon \sup_{0\le t\le T^\varepsilon} \left\| \partial_t^\alpha \partial_{\mathrm{x},\tau,\theta}^\beta V_{app}^\varepsilon(t,\cdot)\right\|_{L^\infty} \le C_{\alpha,\beta}, \quad \forall \alpha, \beta. \tag{2.13}$$

We define

$$U^\varepsilon := \tilde{U} + \varepsilon V_{app}^\varepsilon, \quad \tilde{U} = (U,0)^t, \quad U \in \mathcal{P},$$

and we consider the following equations

$$\begin{cases} \partial_t V + \varepsilon^{-4}\partial_\tau V + \varepsilon^{-1}(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta)(U^\varepsilon, V) + \dfrac{1}{2}(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta)(V,V) \\ - \left(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta\right)^2 V = \left(\left(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta\right) p,\, 0\right)^t + F^\varepsilon, \\ \mathrm{Div}_{\mathrm{x}} V + \varepsilon^{-2}\mathrm{Div}_\theta V = 0, \end{cases} \tag{2.14}$$

We distinguish between the low frequency part V_l and the high frequency part V_h of V. We introduce $\chi = \chi(\zeta, \xi) \in C^\infty\left(\mathbb{R}^3 \times \mathbb{T}^3\right)$ such that

$$\chi(\zeta,\xi) = 1, \quad \text{for } |\zeta + \xi| \le \delta,$$
$$\chi(\zeta,\xi) = 0, \quad \text{for } |\zeta + \xi| \ge 2\delta,$$

where δ is a fixed number satisfying $0 < \delta < 1/4$. Then we set

$$V_l = \chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)V, \quad p_l = \chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)p, \quad V_h = V - V_l, \quad p_h = p - p_l,$$

where for any f, $\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)f$ is the Fourier multiplier defined as

$$\mathcal{F}\left(\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)f\right)(\zeta,\xi) = \chi(\varepsilon^2\zeta, \xi)\mathcal{F}(f)(\zeta,\xi),$$

$\mathcal{F}$ being the Fourier transform with respect to x and θ. Finally, we define for all $s \in N$, for all $t \in [0, T^\varepsilon]$,

$$\Psi_s(V;t) := \|V_l(t)\|^2_{H^s} + \varepsilon^2\|V_h(t)\|^2_{H^s} + \int_0^t \|\left(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta\right)V_l(u)\|^2_{H^s}\,du$$
$$+ \varepsilon^{-2}\int_0^t \|V_h(u)\|^2_{H^s}\,du + \varepsilon^{-2}\int_0^t \|(\varepsilon^2\nabla_{\mathrm{x}} + \nabla_\theta)V_h(u)\|^2_{H^s}\,du,$$

$$\alpha_s(V;t) = \sup_{0\le u\le t} \Psi_s(V;u).$$

We show the following:

Proposition 2.1. *Let $V \in C^0\left([0,T^\varepsilon]; H^\infty\right)^6$, satisfying (2.14). Then, the following inequality holds, for all $s \ge 5$, for ε small enough:*

$$\alpha_s(V;t) \le \mathcal{C}_s\left(\alpha_s(V;0) + \varepsilon^6\int_0^t \|F_h^\varepsilon(u)\|^2_{H^s}\,du + \int_0^t \|F_l^\varepsilon(u)\|^2_{H^s}\,du \right.$$
$$\left. + (1 + \alpha_s(V;t))\int_0^t \alpha_s(V;u)\,du + \alpha_s(V;t)^2\right). \tag{2.15}$$

Proof :
We start with an L^2 estimate on (2.14).

L^2 estimates

- Estimates on V_h

We apply $\left(1 - \chi\left(\varepsilon^2 D_{\mathrm{x}}, D_\theta\right)\right)$ to (2.14). Multiplication by V_h and integration yield

$$\|V_h(t)\|^2_{L^2} + \varepsilon^{-4}\int_0^t \|(\varepsilon^2\nabla_{\mathrm{x}} + \nabla_\theta)V_h\|^2_{L^2} \le \|V_h(0)\|^2_{L^2}$$
$$+ \frac{\|U^\varepsilon\|_\infty}{\varepsilon^3}\int_0^t \|V\|_{L^2}\,\|(\varepsilon^2\nabla_{\mathrm{x}} + \nabla_\theta)V_h\|_{L^2}$$
$$+ \int_0^t \|V\otimes V\|_{L^2}\,\|(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta)V_h\|_{L^2} + \int_0^t \|F_h^\varepsilon\|_{L^2}\,\|V_h\|_{L^2}.$$

Remind that notation $V \otimes V$ refers to the matrix $(V_j\, V_k)_{j,k}$.

By Plancherel formula,

$$\begin{aligned}\|(\varepsilon^2\nabla_{\mathrm{x}} + \nabla_\theta)V_h\|_{L^2}^2 &= \frac{1}{(2\pi)^6}\int |\varepsilon^2\zeta + \xi|^2 |\mathcal{F}(V_h)(\zeta,\xi)|^2 d\zeta d\xi \\ &\geq \frac{\delta^2}{(2\pi)^6}\int |\mathcal{F}(V_h)(\zeta,\xi)|^2 d\zeta d\xi = \delta^2\|V^h\|_{L^2}^2.\end{aligned}$$

Hence, for ε small enough,

$$\begin{aligned}\|V_h(t)\|_{L^2}^2 + \varepsilon^{-4}\int_0^t \|V_h\|_{L^2}^2 + \varepsilon^{-4}\int_0^t \|(\varepsilon^2\nabla_{\mathrm{x}} + \nabla_\theta)V_h\|_{L^2}^2 \leq C\Big(\|V_h(0)\|_{L^2}^2 \\ \varepsilon^{-2}\int_0^t \|V_l\|_{L^2}^2 + \int_0^t \|V\otimes V\|_{L^2}^2 + \varepsilon^4\int_0^t \|F_h^\varepsilon\|_{L^2}^2\Big).\end{aligned} \tag{2.16}$$

- Estimates on V_l

The low frequency part V_l satisfies

$$\begin{aligned}&\partial_t V_l - (\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta))^2 V_l = -\big((\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta)p_l, 0\big)^t \\ &-\frac{1}{2}\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(V,V) - \varepsilon^{-1}\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(U^\varepsilon, V).\end{aligned}$$

The key observation is that

$$\begin{aligned}&\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(U^\varepsilon, V) \\ &\qquad = \chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(\tilde{U}, V_h) \\ &\qquad\qquad + \varepsilon\,\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(V_{app}^\varepsilon, V).\end{aligned}$$

Indeed,

$$\begin{aligned}\operatorname{supp}\mathcal{F}\Big(\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)\big(\tilde{U}, V_l\big)\Big) &\subset \operatorname{supp}\mathcal{F}(U) + \operatorname{supp}\mathcal{F}(V_l) \\ &\subset \bigcup_{\xi'\in\mathbb{Z}^3-\{0\}}\{(0,\xi')\} + \big\{(\zeta,\xi),\ |\varepsilon^2\zeta + \xi| \leq 2\delta\big\} \\ &\subset \big\{(\zeta,\xi), \quad |\varepsilon^2\zeta + \xi| \geq 1 - 2\delta\big\}\end{aligned}$$

As $\delta < 1/4$, we get

$$\operatorname{supp}\mathcal{F}(U) + \operatorname{supp}\mathcal{F}(V_l) \subset \big\{(\zeta,\xi), \quad |\varepsilon^2\zeta + \xi| > 2\delta\big\}$$

so that

$$\chi(\varepsilon^2 D_{\mathrm{x}}, D_\theta)\left(B_{\mathrm{x}} + \varepsilon^{-2}B_\theta\right)(\tilde{U}, V_l) = 0$$

We end with the following energy estimates

$$\begin{aligned}&\|V_l(t)\|_{L^2}^2 + \int_0^t \|(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta)V_l\|_{L^2}^2 \leq C\Big(\|V_l(0)\|_{L^2}^2 \\ &\qquad + \varepsilon^{-2}\int_0^t \|V_h\|_{L^2}^2 + \int_0^t \|V_l\|_{L^2}^2 + \int_0^t \|V\otimes V\|_{L^2}^2 + \int_0^t \|F_l^\varepsilon\|_{L^2}^2\Big).\end{aligned}$$

Combining with (2.16), we get

$$\Psi_0(V;t) \le C\Big(\Psi_0(V;0) + \int_0^t \|V_l\|_{L^2}^2 + \int_0^t \|V \otimes V\|_{L^2}^2 + \varepsilon^6 \int_0^t \|F_h^\varepsilon\|_{L^2}^2 + \int_0^t \|F_l^\varepsilon\|_{L^2}^2\Big).$$

<u>H^s estimates</u>

We do not detail the derivation of the H^s estimates, $s \ge 1$. They follow from differentiating system (2.14), and applying the same argument as above. We get, for all s, for small enough ε:

$$\Psi_s(V;t) \;\le\; C_s \left(\Psi_s(V;0) + \int_0^t \|V_l\|_{H^s}^2 + \int_0^t \|V \otimes V\|_{H^s}^2 + \varepsilon^6 \int_0^t \|F_h^\varepsilon\|_{H^s}^2 + \int_0^t \|F_l^\varepsilon\|_{H^s}^2\right). \quad (2.17)$$

It remains to handle the quadratic terms. As $s \ge 5 > 7/2$, $H^s\left(\mathbb{R}^3 \times \mathbb{T} \times \mathbb{T}^3\right)$ is a Banach algebra. Hence, we obtain the following bounds

$$\begin{aligned}\int_0^t \|V_l \otimes V_h\|_{H^s}^2 &\le \int_0^t \|V_l\|_{H^s}^2 \|V_h\|_{H^s}^2 \\ &\le \left(\sup_{0\le u\le t} \|V_l(u)\|_{H^s}^2\right)\left(\int_0^t \|V_h\|_{H^s}^2\right),\end{aligned}$$

and in the same way,

$$\int_0^t \|V_l \otimes V_l\|_{H^s}^2 \le \left(\sup_{0\le u\le t} \|V_l(u)\|_{H^s}^2\right)\left(\int_0^t \|V_l\|_{H^s}^2\right),$$

$$\int_0^t \|V_h \otimes V_h\|_{H^s}^2 \le \left(\sup_{0\le u\le t} \varepsilon^2 \|V_h(u)\|_{H^s}^2\right)\left(\varepsilon^{-2}\int_0^t \|V_h\|_{L^2}^2\right).$$

If we inject these bounds in (2.17), we obtain

$$\Psi_s(V;t) \le C_s \left(\Psi_s(V;0) + \varepsilon^6 \int_0^t \|F_h^\varepsilon(u)\|_{H^s}^2 du + \int_0^t \|F_l^\varepsilon(u)\|_{H^s}^2 du + (1+\alpha_s(V;t))\int_0^t \Psi_s(V;u)du + \alpha_s(V;t)\Psi_s(V;t)\right). \quad (2.18)$$

Bound (2.15) follows. This ends the proof of the proposition.

2.3. Proof of theorem 1.4. We now turn to the proof of theorem 1.4. Let $\overline{V}_0 = (v_0, b_0)$ in $H^\infty\left(\mathbb{R}^3\right)^6$, such that $\mathrm{Div}_{\mathrm{x}} \overline{V}_0 = 0$. From computations of section 2.1, we infer the existence of $\delta > 0$, such that: if

$$m \ge 1, \quad \text{or} \quad \|\overline{V}_0\|_{H^{1/2}} \le \delta \quad (2.19)$$

there exist approximate solutions of (2.2), indexed by $n \in \mathbb{N}$,

$$V_{app}^{\varepsilon,n} = \varepsilon^m \left(\overline{V}^0(t,\mathrm{x}) + \sum_{i=1}^{n} \varepsilon^i V^i\,(t,\mathrm{x},\tau,\theta) + \varepsilon^{n+1} V_*^{n+1} \right), \tag{2.20}$$

$$p_{app}^{\varepsilon,n} = \varepsilon^m \left(p_*^0(t,\mathrm{x},\tau,\theta) + \sum_{i=1}^{n} \varepsilon^i p^i\,(t,\mathrm{x},\tau,\theta) + \varepsilon^{n+1} \overline{p}^{n+1} \right), \tag{2.21}$$

with $\overline{V}^0|_{t=0} = \overline{V}_0$, and $\overline{V}^i|_{t=0} = 0$ for $i \geq 1$. Profiles V^i and p^i are fund using the recursion of the previous section. They satisfy:

$$\begin{cases} \partial_t V_{app}^{\varepsilon,n} + \varepsilon^{-4} \partial_\tau V_{app}^{\varepsilon,n} + \varepsilon^{-1} \left(B_\mathrm{x} + \varepsilon^{-2} B_\theta \right) (\tilde{U}, V_{app}^{\varepsilon,n}) \\ + \frac{1}{2} \left(B_\mathrm{x} + \varepsilon^{-2} B_\theta \right) (V_{app}^{\varepsilon,n}, V_{app}^{\varepsilon,n}) - \left(\nabla_\mathrm{x} + \varepsilon^{-2} \nabla_\theta \right)^2 V_{app}^{\varepsilon,n} \\ = - \left(\left(\nabla_\mathrm{x} + \varepsilon^{-2} \nabla_\theta \right) p_{app}^{\varepsilon,n},\, 0 \right)^t + R_{app}^{\varepsilon,n}, \\ \left(\mathrm{Div}_\mathrm{x} \,+\, \varepsilon^{-2} \mathrm{Div}_\theta \right) V_{app}^{\varepsilon,n} = r_{app}^{\varepsilon,n}, \end{cases} \tag{2.22}$$

where for all $s, \alpha \in \mathbb{N}$, the remainders $R_{app}^{\varepsilon,n}$ and $r_{app}^{\varepsilon,n}$ satisfy the following estimates:

$$\sup_{0 \leq t \leq T^\varepsilon} \|\partial_t^\alpha R_{app}^{\varepsilon,n}(t)\|_{H^s} \leq C_{\alpha,s}\, \varepsilon^{m+n-2}, \qquad \sup_{0 \leq t \leq T^\varepsilon} \|\partial_t^\alpha r_{app}^{\varepsilon,n}(t)\|_{H^s} \leq C_{\alpha,s}\, \varepsilon^{m+n}. \tag{2.23}$$

We want the second equation of (2.22) to become homogeneous. This is useful in the energy estimates, to get rid of the gradient pressure term. We need to build a function W_{app}^ε, "sufficiently small", such that

$$\left(\mathrm{Div}_\mathrm{x} \,+\, \varepsilon^{-2} \mathrm{Div}_\theta \right) W_{app}^\varepsilon = -r_{app}^{\varepsilon,n}.$$

A natural try would be to look for W_{app}^ε in the form

$$W_{app}^\varepsilon = \begin{pmatrix} \nabla_\mathrm{x} + \varepsilon^{-2} \nabla_\theta & \\ & \nabla_\mathrm{x} + \varepsilon^{-2} \nabla_\theta \end{pmatrix} \Psi^\varepsilon, \tag{2.24}$$

with

$$- \left(\nabla_\mathrm{x} \,+\, \varepsilon^{-2} \nabla_\theta \right)^2 \Psi^\varepsilon = r_{app}^{\varepsilon,n}. \tag{2.25}$$

However, it is not obvious that equation (2.25) has a solution. Indeed, the operator $-\left(\nabla_\mathrm{x} \,+\, \varepsilon^{-2} \nabla_\theta \right)^2$ is not elliptic: its symbol $(\zeta \,+\, \varepsilon^{-2} \xi)^2$ cancels for all $\zeta = -\varepsilon^{-2} \xi, \quad \xi \in \mathbb{Z}$.

To get rid of this difficulty, we have again to distinguish between average and oscillations, low frequencies and high frequencies. We first notice that

$$r_{app}^{\varepsilon,n} = \varepsilon^{m+n} \mathrm{Div}_\mathrm{x} \left(V_*^n + \varepsilon V_*^{n+1} \right) = r_{app,*}^{\varepsilon,n}.$$

Then, we infer from the regularity of $V_{app}^{\varepsilon,n}$ that

$$\begin{aligned} &\|\partial_t^\alpha \partial_\tau^\beta V_{app,*,l}^{\varepsilon,n}(t,\cdot,\tau,\cdot)\|_{H^s}^2 \\ &\leq \frac{1}{(2\pi)^6} \int_{\{|\varepsilon^2 \zeta + \xi| \,\leq\, 2\delta\}} (1 + |\zeta|^2 + |\xi|^2)^s \, \left| \mathcal{F}(\partial_t^\alpha \partial_\tau^\beta V_{app,*}^{\varepsilon,n}(t,\cdot,\tau,\cdot)) \right|^2 \, d\zeta d\xi \\ &\leq \frac{1}{(2\pi)^6} \int_{\{|\zeta| \,\geq\, (1-2\delta)\, \varepsilon^{-2}\}} (1 + |\zeta|^2 + |\xi|^2)^s \, \left| \mathcal{F}(\partial_t^\alpha \partial_\tau^\beta V_{app,*}^{\varepsilon,n}(t,\cdot,\tau,\cdot)) \right|^2 \, d\zeta d\xi \\ &\leq C_k\, \varepsilon^{4k} \, \|\partial_t^\alpha \partial_\tau^\beta V_{app}^{\varepsilon,n}(t,\cdot,\tau,\cdot)\|_{H^{s+k}} = O(\varepsilon^{4k}), \qquad \forall k > 0. \end{aligned} \tag{2.26}$$

Thus, redefining

$$V^{\varepsilon,n}_{app} := \overline{V}^{\varepsilon,n}_{app} \, + \, V^{\varepsilon,n}_{app,*,h},$$

we see that $V^{\varepsilon,n}_{app}$ still satisfies a system of type (2.22), with estimates (2.23), and the new remainder $r^{\varepsilon,n}_{app}$ is such that

$$r^{\varepsilon,n}_{app,l} = 0.$$

In particular, the function W^{ε}_{app} given by (2.24), (2.25) is well-defined, and is bounded by

$$\|\partial^{\alpha}_t W^{\varepsilon}_{app}\|_{H^{s-1}} \, \le \, C_{\alpha,s}\, \varepsilon^2 \, \|\partial^{\alpha}_t r^{\varepsilon}_{app}\|_{H^s}, \quad \forall \alpha, s.$$

Finally, we set $V^{\varepsilon}_{app} = V^{\varepsilon,n}_{app} + W^{\varepsilon}_{app}$. It solves

$$\begin{cases} \partial_t V^{\varepsilon}_{app} + \varepsilon^{-4}\partial_\tau V^{\varepsilon}_{app} + \varepsilon^{-1}\left(B_{\mathrm{x}} + \varepsilon^{-2} B_\theta\right)(\tilde{U}, V^{\varepsilon}_{app}) \\ + \dfrac{1}{2}\left(B_{\mathrm{x}} + \varepsilon^{-2} B_\theta\right)(V^{\varepsilon}_{app}, V^{\varepsilon}_{app}) - \left(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta\right)^2 V^{\varepsilon}_{app} \\ = -\left(\left(\nabla_{\mathrm{x}} + \varepsilon^{-2}\nabla_\theta\right) p^{\varepsilon}_{app}, 0\right)^t + R^{\varepsilon}_{app}, \\ \mathrm{Div}_{\mathrm{x}} V \, + \, \varepsilon^{-2}\mathrm{Div}_\theta V = 0, \end{cases} \tag{2.27}$$

where

$$\sup_{0 \le t \le T^{\varepsilon}} \|R^{\varepsilon}_{app}(t)\|_{H^s} \le C_s \, \varepsilon^{m+n-2}, \quad \forall s. \tag{2.28}$$

We can now use the results of section 2.2. We fix $n \ge 4$, $s \ge 5$, and assume (2.19). For ε small enough, we will show the existence of a solution of (2.2), $V^{\varepsilon} \in C^0\left([0,T];\, H^{\infty}\left(\mathbb{R}^3 \times \mathbb{T} \times \mathbb{T}^3\right)\right)^6$, $V^{\varepsilon}|_{t=0} = \varepsilon^m \overline{V}_0$.

The local existence theory of smooth solutions for (2.2) is classical (see for instance lecture notes [11]). For all $\varepsilon > 0$, there exists a unique maximal solution

$$V^{\varepsilon} \in C^0\left([0, T^{\varepsilon}_*);\, H^{\infty}\left(\mathbb{R}^3 \times \mathbb{T} \times \mathbb{T}^3\right)\right)^6, \quad V^{\varepsilon}|_{t=0} = \varepsilon^m \overline{V}_0.$$

Moreover, the lifespan T^{ε}_* satisfies one of the following conditions:

- $T^{\varepsilon}_* \ge T$
- $T^{\varepsilon}_* < T$ and $\liminf_{t \to T^{\varepsilon}_*} \|V^{\varepsilon}(t)\|_{H^s} \to +\infty$

It is enough to show that the second possibility does not occur for small enough ε. Let $T^{\varepsilon} < \min(T^{\varepsilon}_*, T)$, and define $W^{\varepsilon} := V^{\varepsilon} - V^{\varepsilon}_{app}$ on $[0, T^{\varepsilon}]$. Then, W^{ε} is a solution of (2.14), where V^{ε}_{app} defined above satisfies (2.13), and $F^{\varepsilon} = R^{\varepsilon}_{app}$ satisfies (2.28).

We apply proposition 2.1. It yields, for ε small enough:

$$\alpha_s(W^{\varepsilon}; t) \le C\Big(\varepsilon^{2m+4} + \alpha_s(W^{\varepsilon}; 0) + (1 + \alpha_s(W^{\varepsilon}; t) \int_0^t \alpha_s(W^{\varepsilon}; u)\, du + \alpha_s(W^{\varepsilon}; t)^2\Big). \tag{2.29}$$

Note that $W^\varepsilon|_{t=0} = \varepsilon^m \overline{V}_0 - V^\varepsilon_{app}|_{t=0}$ satisfies

$$\overline{W}^\varepsilon|_{t=0} = 0, \quad \|W^\varepsilon_*|_{t=0}\|_{H^s} = O(\varepsilon^{m+1}).$$

As in (2.26), we deduce that

$$\|W^\varepsilon_l|_{t=0}\|_{H^s} = O(\varepsilon^k), \quad \forall k, \tag{2.30}$$

and that

$$\alpha_s(W^\varepsilon; 0) = O(\varepsilon^{2m+4}).$$

We introduce

$$T(\varepsilon) := \sup\left\{t \in [0, T^\varepsilon), \quad \alpha_s(W^\varepsilon; t) < \varepsilon^{2m+3}\right\}.$$

For $\varepsilon > 0$ small enough, $T(\varepsilon)$ is well-defined and positive. Moreover, using (2.29) : $\forall t < T(\varepsilon)$,

$$(1 - \varepsilon^{2m+3})\, \alpha_s(W^\varepsilon; t) \le C\Big(\varepsilon^{2m+4} \; + \; (1 + \varepsilon^{2m+3}) \int_0^t \alpha_s(W^\varepsilon; u) du\Big),$$

The Gronwall's lemma implies: $\forall t < T(\varepsilon)$, $\alpha_s(W^\varepsilon; t) \le C'\, \varepsilon^{2m+4}$. This last inequality shows that $T(\varepsilon) = T^\varepsilon$, and that

$$\sup_{0 \le t \le T^\varepsilon} \alpha_s(W^\varepsilon; t) \; \le \; C'' \varepsilon^{2m+4}.$$

In particular, we get

$$\sup_{0 \le t \le T^\varepsilon} \|W^\varepsilon(t, \cdot)\|_{H^s} \le C\varepsilon^{m+1}.$$

Back to $V^\varepsilon = V^\varepsilon_{app} + W^\varepsilon$, we obtain, for all s

$$\begin{aligned} \sup_{0 \le t \le T^\varepsilon} \|V^\varepsilon(t, \cdot)\|_{H^s} &\le \sup_{0 \le t \le T^\varepsilon} \|V^\varepsilon_{app}(t, \cdot)\|_{H^s} + \sup_{0 \le t \le T^\varepsilon} \|W^\varepsilon(t, \cdot)\|_{H^s} \\ &\le \; C\varepsilon^m \|\overline{V}_0\|_{H^s}. \end{aligned} \tag{2.31}$$

This yields $T^\varepsilon_* \ge T$, and shows the existence on $[0, T]$ of a smooth solution V^ε with initial data $\overline{V}_0$.

As explained at the beginning of section 2, this provides a smooth solution

$$(v^\varepsilon, b^\varepsilon)^t(t, \mathrm{x}) = V^\varepsilon\left(t, \mathrm{x}, \varepsilon^{-4} t, \varepsilon^{-2} \mathrm{x}\right)$$

of (1.5), with initial data $(\varepsilon^m v_0, \varepsilon^m b_0)^t$. Uniqueness of the solution $(v^\varepsilon, b^\varepsilon)$ is a direct consequence of its regularity. Finally, the estimates of the theorem follow from (2.31). It ends the proof.

3. Instability mechanism. We now begin the description of the instability mechanism, leading to theorem 1.5. As we will see, this mechanism is connected to the behaviour of the WKB solutions (2.3).

3.1. Spectral analysis. To understand the instability process requires the study of system (2.8), satisfied by $V^0 = \overline{V}^0(t,\mathrm{x})$ when $m \geq 1$. This system reads

$$\begin{cases} \partial_t \overline{w}^0 = \mathcal{A}_1 \overline{w}^0 + \nabla_{\mathrm{x}} \overline{p}^1 + \Delta_{\mathrm{x}} \overline{w}^0, \\ \partial_t \overline{\beta}^0 = \mathcal{A}_2 \overline{\beta}^0 + \Delta_{\mathrm{x}} \overline{\beta}^0, \\ \mathrm{div}_{\mathrm{x}} \overline{w}^0 = \mathrm{div}_{\mathrm{x}} \overline{\beta}^0 = 0, \end{cases} \tag{3.1}$$

where the operators

$$\mathcal{A}_1 \overline{w} = \mathrm{div}_{\mathrm{x}}(A_1 \overline{w}), \quad \mathcal{A}_2 \overline{\beta} = \mathrm{curl}_{\mathrm{x}}(A_2 \overline{\beta}),$$

involve the linear operators $A_1 \in \mathcal{L}\left(\mathbb{R}^3, \mathcal{M}_3(\mathbb{R})\right)$, $A_2 \in \mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^3\right)$. They are defined in the following way: for all $b \in \mathbb{R}^3$,

$$\begin{aligned} A_1 b &= -\int_{\tau,\theta} \left((\partial_\tau - \Delta_\theta)^{-1} \left(\nabla_\theta \Delta_\theta^{-1} \mathrm{div}_\theta^2 (U \otimes b) \; - \; \mathrm{div}_\theta\, (U \otimes b) \right) \right) \otimes U, \\ A_2 b &= \int_{\tau,\theta} U \times \left((\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta (U \times b) \right). \end{aligned}$$

As

$$\mathrm{div}_\theta (U \otimes b) = b \cdot \nabla_\theta U = \mathrm{curl}_\theta (U \times b),$$

we deduce $\mathrm{div}_\theta^2 (U \otimes b) = 0$ and

$$A_1 b = \int_{\tau,\theta} \otimes \left((\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta\, (U \times b) \right) \otimes U.$$

Then,

$$\begin{aligned} \mathcal{A}_1 a &= -\mathrm{div}_{\mathrm{x}} \int_{\tau,\theta} \left((\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta\, (U \times b) \right) \otimes U \\ &= \int_{\tau,\theta} U \cdot \nabla_{\mathrm{x}} (\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta (U \times b) = -\mathcal{A}_2 b. \end{aligned}$$

If we re-label for all $b \in \mathbb{R}^3$,

$$A b = \int_{\tau,\theta} U \times \left((\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta (U \times b) \right),$$

the system on $\overline{V}^0$ reads eventually

$$\begin{cases} \partial_t \overline{w}^0 = -\mathrm{curl}_{\mathrm{x}} A\, \overline{w}^0 + \nabla_{\mathrm{x}} \overline{p}^1 + \Delta_{\mathrm{x}} \overline{w}^0, \\ \partial_t \overline{\beta}^0 = \mathrm{curl}_{\mathrm{x}} A\, \overline{\beta}^0 + \Delta_{\mathrm{x}} \overline{\beta}^0, \\ \mathrm{div}_{\mathrm{x}} \overline{w}^0 = \mathrm{div}_{\mathrm{x}} \overline{\beta}^0 = 0. \end{cases} \tag{3.2}$$

REMARK 3.1. *In the physical litterature, the matrix A is often denoted α, which motivates the expression* alpha-effect*.*

We focus on solutions of (3.2) with initial data $(0, \overline{\beta}_0)^t$, $\mathrm{div_x}\, \overline{\beta}_0 = 0$. Thus, $\overline{w}_0 = 0$ for all times, and the divergence free condition $\mathrm{div_x}\overline{\beta}^0 = 0$ is fulfilled for all times. If we set $b^0 := \overline{\beta}^0$, we are left with the study of

$$\partial_t b^0 - \mathrm{curl_x}\left(A b^0\right) - \Delta_{\mathrm{x}} b^0 = 0. \tag{3.3}$$

We first state some properties of the matrix A.

LEMMA 3.2. *For all $U \in \mathcal{P}$, the matrix $A = A(U)$ is real symmetric. Moreover, the set*

$$\Omega = \{U \in \mathcal{P}, \quad A(U) \ \textit{has simple non-zero eigenvalues}\}$$

is dense and open in $\mathcal{P}$. **Proof :** Using the Fourier coefficients of U, we compute for all $b \in \mathbb{R}^3$:

$$A\, b = \sum_{\substack{(\omega,k)\in\mathbb{Z}^4\\(\omega,k)\neq 0}} \frac{\hat{U}(\omega,k) \times \hat{U}(-\omega,-k)\,(-ib\cdot k)}{i\omega + |k|^2}. \tag{3.4}$$

Remark that the change of indices $(\omega, k) \to (\omega' = -\omega, k' = -k)$ yields

$$\begin{aligned} A\, b &= \sum_{(\omega',k')} \frac{\hat{U}(-\omega',-k') \times \hat{U}(\omega',k')\,(ib\cdot k')}{-i\omega' + |k'|^2} \\ &= \sum_{(\omega',k')} \frac{\hat{U}(\omega',k') \times \hat{U}(-\omega',-k')\,(-ib\cdot k')}{-i\omega' + |k'|^2}, \end{aligned}$$

so that

$$A\, b = \int_{\mathbb{T}^4} U \times (-\partial_\tau - \Delta_\theta)^{-1} \mathrm{curl}_\theta\,(U \times b)\,. \tag{3.5}$$

As U is real-valued, $\hat{U}(-\omega,-k) = \hat{U}^*(\omega,k)$, where the star denotes complex conjugation. We deduce, for all $b \in \mathbb{R}^3$,

$$\begin{aligned} A^*\, b &= \sum_{(\omega,k)} \frac{\hat{U}^*(\omega,k) \times \hat{U}^*(-\omega,-k)\,(ib\cdot k)}{-i\omega + |k|^2} \\ &= \sum_{(\omega,k)} \frac{\hat{U}(-\omega,-k) \times \hat{U}(\omega,k)\,(ib\cdot k)}{-i\omega + |k|^2} \\ &= \sum_{(\omega',k')} \frac{\hat{U}(\omega',k') \times \hat{U}(-\omega',-k')\,(-ib\cdot k')}{i\omega' + |k'|^2} = \alpha\, b \end{aligned}$$

so that α is real. Then, we compute

$$\begin{aligned}
A\,b\cdot\tilde b &= \int_{\mathbb{T}^4}\Big(U\times(\partial_\tau-\Delta_\theta)^{-1}\mathrm{curl}_\theta\,(U\times b)\Big)\cdot\tilde b\\
&= -\int_{\mathbb{T}^4}\Big(U\times\tilde b\Big)\cdot\Big((\partial_\tau-\Delta_\theta)^{-1}\mathrm{curl}_\theta\,(U\times b)\Big)\\
&= -\int_{\mathbb{T}^4}\Big(U\times\tilde b\Big)\cdot\mathrm{curl}_\theta\,(\partial_\tau-\Delta_\theta)^{-1}\,(U\times b)\Big)\\
&= -\int_{\mathbb{T}^4}\Big((-\partial_\tau-\Delta_\theta)^{-1}\mathrm{curl}_\theta\,\Big(U\times\tilde b\Big)\Big)\cdot\Big(U\times b\Big)\\
&= \int_{\mathbb{T}^4}\Big(U\times(-\partial_\tau-\Delta_\theta)^{-1}\mathrm{curl}_\theta\,\Big(U\times\tilde b\Big)\Big)\cdot b = \alpha\,\tilde b\cdot b
\end{aligned}$$

using (3.5). Thus, A is symmetric.

From (3.4), we deduce that

$$\mathcal{P}\mapsto\mathcal{M}_3\,(\mathbb{R})\,,\quad U\mapsto A(U)$$

is continuous. This clearly implies that Ω is open in $\mathcal{P}$. Let $\varepsilon>0$, and $U\in\mathcal{P}-\Omega$. Let

$$U^n(\tau,\theta)=\sum_{|\omega|+|k|\le n}\hat U(\omega,k)\,e^{i(\omega\tau+k\cdot\theta)}.$$

There exists N, such that $d_{\mathcal{P}}\left(U,U^N\right)<\varepsilon/2$. If $U^N\in\Omega$, we are done. Otherwise, we consider

$$\tilde U = U^N\;+\;\sum_{i=1}^{3}\delta_i\,V^i,\quad \delta_i>0 \text{ for all } i,$$

where

$$\begin{aligned}
V^1(\theta_1,\theta_2,\theta_3) &= \Big(\cos((N+1)\theta_2),\sin((N+1)\theta_1),\\
&\qquad \cos((N+1)\theta_1)+\sin((N+1)\theta_2)\Big)^t,\\
V^2(\theta_1,\theta_2,\theta_3) &= \Big(\sin((N+2)\theta_3),\sin((N+2)\theta_1)+\cos((N+2)\theta_3),\\
&\qquad \cos((N+2)\theta_1)\Big)^t,\\
V^3(\theta_1,\theta_2,\theta_3) &= \Big(\sin((N+3)\theta_3)+\cos((N+3)\theta_2),\cos((N+3)\theta_3),\\
&\qquad \sin((N+3)\theta_2)\Big)^t,
\end{aligned}$$

Note that the V^i's are special cases of the famous ABC flows ([1]) of the type:

$$\begin{aligned}
V(\theta_1,\theta_2,\theta_3) = \Big(&C\sin(M\theta_3)+B\cos(M\theta_2),A\sin(M\theta_1)+C\cos(M\theta_3),\\
&B\sin(M\theta_2)+A\cos(M\theta_1)\Big)^t.
\end{aligned}$$

They satisfy the Beltrami property that $\operatorname{curl} u = k\, u$, $k > 0$. A simple calculation shows that

$$A(\tilde{U}) = A(U^N) - \delta_1 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} - \delta_2 \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} - \delta_3 \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Thus, for appropriate choices of δ_1, δ_2, δ_3, one has,

$$A(\tilde{U}) \in \Omega, \quad d_{\mathcal{P}}\left(U^n, \tilde{U}\right) < \varepsilon/2,$$

so that $d_{\mathcal{P}}\left(U, \tilde{U}\right) < \varepsilon$. Thus, Ω is dense, which ends the proof of the lemma.

We can now perform a spectral analysis of equation (3.3).

Proposition 3.3. *Let Ω the subset of $\mathcal{P}$ defined in lemma 3.2. For all U in Ω, there exists $\zeta^0 \in \mathbb{R}^3$, $\delta > 0$, and two smooth functions*

$$\Lambda_+ : B(\zeta^0, \delta) \mapsto \mathbb{R}_*^+, \quad \hat{b} : B(\zeta^0, \delta) \mapsto (\mathbb{C}_*)^3,$$

such that for all $\zeta \in B(\zeta^0, \delta)$,

$$b^\zeta(t, \mathrm{x}) \;=\; \hat{b}(\zeta)\, \exp\left(\Lambda_+(\zeta)\, t\right)\, \exp\left(i\zeta \cdot \mathrm{x}\right)$$

is a divergence-free solution of (3.3)*. Moreover, one can assume that Λ has a non-degenerate maximum over $B(\zeta^0, \delta)$ at ζ^0.* **Proof :** Let U in Ω. We apply the Fourier transform to (3.3): we get, for all $\zeta \in \mathbb{R}^3$,

$$\partial_t \mathcal{F}\left(b^0\right)(t, \zeta) = A^\zeta A\, \mathcal{F}\left(b^0\right)(t, \zeta) - |\zeta|^2 \mathcal{F}\left(b^0\right)(t, \zeta),$$

where

$$\forall \zeta \in \mathbb{R}^3, \quad A^\zeta = \begin{pmatrix} 0 & -i\zeta_3 & i\zeta_2 \\ i\zeta_3 & 0 & -i\zeta_1 \\ -i\zeta_2 & i\zeta_1 & 0 \end{pmatrix}$$

is the matrix corresponding to cross product by $i\zeta$. As A is real-symmetric, there exists an orthogonal matrix P with $P^t A\, P = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{R}$. Introducing $\xi = P^t \zeta$, $\tilde{b}(t, \xi) = P^t \mathcal{F}(b^0)(t, \zeta)$, the previous equation reads

$$\partial_t\, \tilde{b}(t, \xi) = A^\xi D\, \tilde{b}(t, \xi) - |\xi|^2\, \tilde{b}(t, \xi). \tag{3.6}$$

A rapid calculation shows that the eigenvalues λ of $A^\xi D - |\xi|^2 I_d$ satisfy $\lambda = -|\xi|^2$ or

$$\left(\lambda + |\xi|^2\right)^2 \;=\; \xi_1^2 \alpha_2 \alpha_3 \;+\; \xi_2^2 \alpha_1 \alpha_3 \;+\; \xi_3^2 \alpha_1 \alpha_2. \tag{3.7}$$

As $U \in \Omega$, the α_i's are distinct and non-zero. Consequently, the products $\alpha_i \alpha_j$, $i \neq j$ are also distinct and non-zero. Let

$$f(\xi) := \xi_1^2 \alpha_2 \alpha_3 \;+\; \xi_2^2 \alpha_1 \alpha_3 \;+\; \xi_3^2 \alpha_1 \alpha_2, \quad \mathcal{U} := \{\xi,\; f(\xi) > 0\}.$$

There are two possibilities:

- All the α_i's have the same sign. Then, all the $\alpha_i \alpha_j$, $i \neq j$ are positive, and $\mathcal{U}$ is $\mathbb{R}_*^3$.

- The α_i's have different signs. Then, among the products $\alpha_i\alpha_j$, $i \neq j$, two are negative, and one is positive, for instance $\alpha_2\alpha_3$. In this case, $\mathcal{U}$ is the cone $\left\{|\xi_1|^2 > -\frac{\alpha_1\alpha_3}{\alpha_2\alpha_3}|\xi_2|^2 - \frac{\alpha_1\alpha_2}{\alpha_2\alpha_3}|\xi_3|^2\right\}$.

On $\mathcal{U}$, one can define

$$\lambda_\pm(\xi) = \pm f(\xi)^{1/2} \; - \; |\xi|^2.$$

Note that $\lambda_+(\xi)$ takes positive values for some ξ. Indeed, if $\overline{\xi}$ is such that $f(\overline{\xi}) > 0$, then

$$\lambda_+\left(\delta\overline{\xi}\right) = \delta f\left(\overline{\xi}\right)^{1/2} \; - \; \delta^2 \left|\overline{\xi}\right|^2$$

is positive for $\delta > 0$ small enough. Moreover, using $\lambda_+(\xi) \xrightarrow{|\xi|\rightarrow+\infty} -\infty$, we deduce that λ_+ has a global positive maximum in $\mathcal{U}$, say at $\xi = \xi^0$. As ξ^0 is a critical point, we obtain

$$\begin{aligned} 0 = \lambda_+'(\xi^0) &= \frac{1}{2}f(\xi^0)^{-1/2}\nabla f(\xi^0) \; - \; 2\xi^0 \\ &= f(\xi^0)^{-1/2}\begin{pmatrix} \alpha_2\alpha_3 & & \\ & \alpha_1\alpha_3 & \\ & & \alpha_1\alpha_2 \end{pmatrix}\xi^0 - 2\xi^0. \end{aligned}$$

Up to reindex the eigenvalues, this implies that:

$$2f(\xi^0)^{1/2} = \alpha_2\alpha_3, \quad \xi^0 = (\xi^0_1, 0, 0)^t. \tag{3.8}$$

Then, we compute

$$\lambda_+''(\xi^0) = -\frac{1}{2}f(\xi^0)^{-3/2}\nabla f(\xi^0) \otimes \nabla f(\xi^0) + \frac{1}{2}f(\xi^0)^{-1/2}f''(\xi^0) - 2I_d,$$

and using (3.8) leads to

$$\lambda_+''(\xi^0) = \begin{pmatrix} -\frac{1}{2f(\xi^0)^{3/2}}(\xi^0_1)^2 & & \\ & \frac{\alpha_1\alpha_3}{f(\xi^0)^{1/2}} - 2 & \\ & & \frac{\alpha_1\alpha_2}{f(\xi^0)^{1/2}} - 2 \end{pmatrix}.$$

In particular, $\det(\lambda_+''(\xi^0)) \neq 0$, which means that the maximum at ξ^0 is non-degenerate.

Let $\delta > 0$ such that $B(\xi^0, \delta) \subset \mathcal{U}$. For all $\xi \in B(\xi^0, \delta)$, $\lambda_+(\xi)$ has multiplicity one as an eigenvalue of $A^\xi D$. Therefore, classical smooth dependence results on eigenvalues and eigenvectors yield the existence of a smooth function $\tilde{b} : B(\xi^0, \delta) \mapsto (\mathbb{C}_*)^3$ such that

$$A^\xi D\,\tilde{b}(\xi) = \lambda_+(\xi)\,\tilde{b}(\xi).$$

Back to original variables, we define

$$\zeta^0 = P\xi^0, \quad \Lambda_\pm(\zeta) = \lambda_\pm(P^t\zeta), \quad \hat{b}(\zeta) = P\tilde{b}(P^t\zeta),$$

so that

$$\left(A^\xi A - |\zeta|^2\right)\hat{b}(\zeta) = \Lambda_+(\zeta)\,\hat{b}(\zeta).$$

This shows that, for all $\zeta \in B(\zeta^0)$,

$$b^\zeta(t, \mathrm{x}) \; = \; \hat{b}(\zeta)\, e^{i\Lambda(\zeta)t}\, e^{i\zeta\cdot\mathrm{x}}.$$

solves (3.3). Note also that b^ζ is divergence free, as $\zeta \cdot \hat{b}(\zeta) = 0$. This ends the proof of the proposition.

3.2. Construction of unstable wavepackets. Thanks to proposition 3.3, we are now able to build approximate solutions having exponential growth. More precisely, we show

PROPOSITION 3.4. *Let $U \in \Omega$. There exists $\lambda^0 > 0$, and for all integer $n \in \mathbb{N}$ families $\left\{X^i = \left(\overline{V}^i, V_*^{i+1}, p_*^i, \overline{p}^{i+1}\right)\right\}_{0 \le i \le n}$ satisfying:*

i) *For all i, X^i satisfies (T_i).*

ii) *As $t \to +\infty$, $V^0 = \overline{V}^0 = (0, \overline{\beta}^0)$ has the asymptotic behaviour:*

$$\|\overline{\beta}^0(t,\cdot)\|_{L^2} \sim \frac{C}{\sqrt{t}} e^{\lambda^0 t}. \tag{3.9}$$

iii) *For all $i = km + l$, with $l \in \{0, \dots, m-1\}$, for all $\alpha, s \in \mathbb{N}$, for all t,*

$$\|\partial_t^\alpha X^i(t,\cdot)\|_{H^s} \le \frac{C_{\alpha,i,s}}{\sqrt{1+t}^{k+1}}\, t^l\, e^{(k+1)\lambda^0 t}, \quad C_{\alpha,i,s} > 0. \tag{3.10}$$

Proof : We treat separately the case $i = 0,\ 1 \le i \le m-1$, and $i \ge m$ (for which (T_i) includes quadratic terms).

<u>Construction of X^0</u>

We first use proposition 3.3 to build X^0. Let

$$\Lambda_\pm : B(\zeta^0, \delta) \mapsto \mathbb{R}_*^+, \quad \hat{b} : B(\zeta^0, \delta) \mapsto (\mathbb{C}_*)^3$$

as in proposition 3.4. As $\Lambda^0 := \Lambda_+(\zeta^0)$ is a non-degenerate maximum, one can assume, up to take a smaller δ:

$$\Lambda_+(\zeta) = \Lambda_+(\zeta^0) + \nabla_\zeta \Lambda_+(\zeta^0)\cdot(\zeta - \zeta^0) - |\zeta - \zeta^0|^2 \frac{\alpha(\zeta - \zeta^0)}{2}, \tag{3.11}$$

where $0 < \underline{\alpha} < \alpha(\cdot) < \overline{\alpha}$ in $B(0,\delta)$. We extend functions $\Lambda_\pm$ and $\hat{b}$ to

$$B_\delta := B(\zeta^0, \delta) \cup B(-\zeta^0, \delta)$$

by: for all $\zeta \in B(-\zeta^0, \delta)$,

$$\Lambda_\pm(\zeta) := \Lambda_\pm(-\zeta), \quad \hat{b}(\zeta) := \hat{b}(-\zeta)^*.$$

With this continuation,

$$b^\zeta(t, \mathrm{x}) = \hat{b}(\zeta)\, e^{\Lambda_+(\zeta)t}\, e^{i\zeta\cdot \mathrm{x}} \tag{3.12}$$

is a divergence-free solution of (3.3) for all ζ in B_δ. Let now ϕ a smooth real-valued function supported in B_δ, such that $\phi(\zeta^0) = 1$, $\phi(-\zeta) = \phi(\zeta)^*$. We set

$$\overline{\beta}^0(t, \mathrm{x}) = \int_{B_\delta} \phi(\zeta)\, \hat{b}(\zeta)\, e^{\Lambda_+(\zeta)t}\, e^{i\zeta\cdot \mathrm{x}}\, d\zeta. \tag{3.13}$$

Then, we define $\overline{V}^0 := (0, \overline{\beta}^0)^t$, V_*^1 by (2.6), and $X^0 = \left(\overline{V}^0, V_*^1, 0, 0\right)$. From the properties of b^ζ, we deduce easily that $\overline{V}^0$ satisfies (3.2), and consequently that X^0 solves (T_0). Finally, points ii) and iii) of the proposition are derived from standard computations involving (3.11). For the sake of brevity, we do not detail these computations, and refer to [4] for complete treatment in a very similar framework.

Construction of X^i, $1 \le i \le m-1$

Let X^0 as above, and for $1 \le i \le m-1$, let X^i defined inductively by: *X^i is the solution of (T_i) with $\overline{V}^i|_{t=0} = 0$.* As seen in section 3.1, such definition makes sense. We show by induction on $i \ge 0$ the following property:

(P_i)*: Function X^i has an expression of the type:*

$$X^i(t, \mathrm{x}, \tau, \theta) = \int_{B_\delta} \left(P^{i,+}_{\zeta,\tau,\theta}(t)\, e^{\Lambda_+(\zeta)t} + P^{i,-}_{\zeta,\tau,\theta}(t)\, e^{\Lambda_-(\zeta)t} + P^{i,-}_{\zeta,\tau,\theta}(t)\, e^{-|\zeta|^2 t} \right) e^{i\zeta\cdot \mathrm{x}} d\zeta, \quad (3.14)$$

where $P^{i,\pm}_{\zeta,\tau,\theta}$, $P^{i,0}_{\zeta,\tau,\theta}$ are polynomials in t, of degree $\le i$, with coefficients smooth and compactly supported in $B_\delta \times \mathbb{T} \times \mathbb{T}^3$.

- Case $i = 0$

(P_0) is true by definition of X^0.

- Case $i \ge 1$

Let $i \ge 1$, and assume (P_0), ..., (P_{i-1}). Remind that

$$\left(p^i_*, 0\right)^t = \Delta_\theta^{-1} \mathrm{Div}_\theta B_\theta \left(\tilde{U}, \overline{V}^i\right) + H^i_*,$$
$$V^{i+1}_* = (\partial_\tau - \Delta)^{-1} L_\theta \overline{V}^i + I^i_*,$$

where H^i_*, I^i_* are $i-$linear in $(X^0, \dots, X^{i-1})$. Thus, X^i is of type (3.14) as soon as $\overline{V}^i$ and $\overline{p}^{i+1}$ are.

These functions satisfy:

$$\begin{cases} \partial_t \overline{V}^i = \begin{pmatrix} & -\mathrm{curl}_\mathrm{x} A \\ \mathrm{curl}_\mathrm{x} A & \end{pmatrix} V^i + \Delta_\mathrm{x} V^i + \left(\nabla_\mathrm{x} \overline{p^{i+1}}, 0\right)^t + \overline{J}^i, \\ \mathrm{Div}_\mathrm{x} V^i = 0. \end{cases} \quad (3.15)$$

Taking the curl of the first line of (3.15) yields

$$\overline{p}^{i+1} = -(\Delta_\mathrm{x})^{-1}\, \overline{J}^i_1,$$

which shows that

$$\overline{p}^{i+1}(t, \mathrm{x}) = \int_{B_\delta} \left(Q^{i,+}_\zeta(t)\, e^{\Lambda_+(\zeta)t} + Q^{i,-}_\zeta(t)\, e^{\Lambda_-(\zeta)t} + Q^{i,0}_\zeta(t)\, e^{-|\zeta|^2 t} \right) e^{i\zeta\cdot x} d\zeta,$$

where $Q^{i,\pm}_\zeta$, $Q^{i,0}_\zeta$ are polynomials in t, of degree $\le i-1$, with coefficients smooth and compactly supported in B_δ.

Replacing $\overline{p}^{i+1}$ by its expression leads to

$$\begin{cases} \partial_t \overline{V}^i = \begin{pmatrix} & -\mathrm{curl}_\mathrm{x} A \\ \mathrm{curl}_\mathrm{x} A & \end{pmatrix} \overline{V}^i + \Delta_\mathrm{x} \overline{V}^i + \overline{K}^i, \\ \mathrm{Div}_\mathrm{x} \overline{V}^i = 0, \end{cases}$$

where $\overline{K}^i$ is $i-$linear on $(X^0, \dots, X^{i-1})$. We define

$$\mathcal{V}^i(t, \zeta) := \mathcal{F}\left(\overline{V}^i\right)(t, \zeta),$$

which satisfies

$$\partial_t \mathcal{V}^i(t,\zeta) = \begin{pmatrix} -A^\zeta A & \\ & A^\zeta A \end{pmatrix} - |\xi|^2 \mathcal{V}^i(t,\zeta) + \mathcal{K}^i(t,\zeta),$$

$$\mathcal{K}^i(t,\zeta) = R^{i,+}_\zeta(t)\, e^{\Lambda_+(\zeta)t} + R^{i,-}_\zeta(t)\, e^{\Lambda_-(\zeta)t} + R^{i,0}_\zeta(t)\, e^{-|\zeta|^2 t},$$

where $R^{i,\pm}_\zeta$, $R^{i,0}_\zeta$ are polynomials in t of degree $\leq i-1$, smooth and compactly supported in B_δ.

At fixed ζ, such equation is an ordinary differential system, of the type:

$$\frac{d}{dt}\mathcal{V} + M\mathcal{V} = S^+(t)e^{\Lambda_+ t} + S^-(t)e^{\Lambda_- t} + S^0(t)e^{\Lambda^0 t},$$

where Λ_+, Λ_-, Λ_0, which stand for $\Lambda_+(\zeta)$, $\Lambda_-(\zeta)$, $-|\zeta|^2$, are simple eigenvalues of the matrix M, which stands for $\begin{pmatrix} -A^\zeta A & \\ & A^\zeta A \end{pmatrix}$. It is well-known that the solution

$$\mathcal{V}(t) = \int_0^t e^{M(t-s)} \left(S^+(s)e^{\Lambda^+ s} + S^-(s)e^{\Lambda^- s} + S^0(s)e^{\Lambda^0 s} \right) ds$$

with $\mathcal{V}(0) = 0$ is of the type

$$\mathcal{V}(t) = T^+(t)e^{\Lambda^+ t} + T^-(t)e^{\Lambda^- t} + T^0(t)e^{\Lambda^0 t},$$

where $T^\pm$ (resp.T^0) is a polynomial such that $\deg T^\pm \leq \deg S^\pm + 1$ (resp. $\deg T^0 \leq \deg S^0 + 1$.

Back to the original system, (P_i) follows, which ends the induction. Point iii) is again a classical consequence of expression (3.14) and (3.11).

Construction of X^i, $i \geq m$

As $i \geq m$, quadratic terms enter system (T_i). More precisely, one checks that

- H^i_* and I^i_* are made of terms that are $i-$linear in $(X^0, \dots, X^{i-1})$, and of quadratic terms that involve the pairs $\{V^j, V^J\}$ with $j + J = i - m - 1$, or $i - m - 3$.
- $\overline{J}^i$ and $\overline{K}^i$ are made of terms that are i-linear in $(X^0, \dots, X^{i-1})$, and of quadratic terms involving the pairs $\{V^j, V^J\}$ with $j + J = i - m$, $i - m - 1$ or $i - m - 3$.

Note that

$$\Lambda_+(j\zeta^0) \;<\; j\,\Lambda_+(\zeta^0), \quad \forall 2 \leq j \leq n,$$

so that up to take δ smaller, one can assume: for all $2 \leq j \leq n$ and for all ζ_1,..., ζ_j in B_δ,

$$\Lambda_+(\zeta_1 + \ldots + \zeta_j) \;<\; \Lambda_+(\zeta_1) \,+ \ldots +\, \Lambda_+(\zeta_j).$$

Under this assumption, one can show inductively that for general $i = km + l$, $k \geq 0$, $1 \leq l \leq m-1$, the solution X^i of (T_i) with $\overline{V}^i|_{t=0} = 0$ has an expression of the type:

$$X^i(t,\mathrm{x},\tau,\theta) = \sum_{j=1}^{k+1} \int_{B_\delta^{j+1}} Y^{i,j}(\zeta_1,\dots,\zeta_j,\tau,\theta,t)\, e^{i(\zeta_1+\ldots+\zeta_j)\cdot \mathrm{x}}\, d\zeta_1 \ldots d\zeta_j +$$

$$\int_{B_\delta^{k+1}} P^i(\zeta_1,\dots,\zeta_{k+1},\tau,\theta,t)\, e^{(\Lambda_+(\zeta_1)+\ldots+\Lambda_+(\zeta_{k+1}))\,t}\, e^{i(\zeta_1+\ldots+\zeta_{k+1})\cdot \mathrm{x}}\, d\zeta_1 \ldots d\zeta_{k+1},$$

where

- P^i is polynomial in t, of degree $\le l$, with coefficients smooth and compactly supported.
- $Y^{i,j}$ is a finite sum of terms of the form

$$Y^i_{\Lambda_1,\dots,\Lambda_j}(\zeta_1,\dots,\zeta_j,\tau,\theta,t) = Q^i_{\Lambda_1,\dots,\Lambda_j}(\zeta_1,\dots,\zeta_j,\tau,\theta,t)\, e^{(\Lambda_1(\zeta_1)+\dots+\Lambda_j(\zeta_{k+1}))\,t},$$

with $Q^i_{\Lambda_1,\dots,\Lambda_j}$ polynomial in t, and $\Lambda_1(\zeta_0)+\dots\Lambda_j(\zeta_0) < (k+1)\Lambda_+(\zeta^0)$

We do not detail this induction, as it is very close from the previous one and tedious. Once this expression for X^i obtained, the estimate follows, see again [4] for all details.

3.3. Proof of theorem 1.5. We now turn to the proof of the instability theorem 1.5. We adapt ideas of [9], encountered in the stability study of Euler and and Prandtl equations (see also [6, 12]).

Let k_0 in $\mathbb{N}^*$ to be chosen later, $m \ge 1$, and $n = k_0 m$. Let $U \in \Omega$, and take profiles $X^0, \dots, X^n$ as in proposition 3.4. We have in particular

$$\varepsilon^m \|\overline{V}^0(t,\cdot)\|_{L^2} = \varepsilon^m \|\overline{\beta}^0(t,\cdot)\|_{L^2} \ge \frac{C_0\,\varepsilon^m\, e^{\lambda^0 t}}{(1+t)^{1/2}}.$$

We define

$$\mathcal{E}(t) := \frac{C_0\,\varepsilon^m\, e^{\lambda^0 t}}{(1+t)^{1/2}},$$

and $T^\varepsilon > 0$ such that $\mathcal{E}(T^\varepsilon) = 1$. Note that $T^\varepsilon = O(|\ln(\varepsilon)|)$. Moreover, thanks to point iii) of proposition (3.4), we get: for all $i = km + l$, $l \in \{0,\dots,m-1\}$, for all α, s, t,

$$\begin{aligned} \varepsilon^{m+i}\|\partial_t^\alpha X^i(t)\|_{H^s} &\le C\,\frac{\varepsilon^{m+i} e^{(k+1)\lambda_0 t}}{\sqrt{1+t}^{k+1}}\, t^l, \\ &\le C\left(\frac{\varepsilon^m e^{\lambda_0 t}}{\sqrt{1+t}}\right)^{k+1} (\varepsilon t)^l, \\ &\le C'\,\mathcal{E}(t)^{k+1}\,(\varepsilon t)^l \end{aligned} \tag{3.16}$$

As in section 2.3, we define $V^{\varepsilon,n}_{app}$ and $p^{\varepsilon,n}_{app}$ by (2.3), so that they satisfy equations of type (2.14). Thanks to (3.16), it is easy to check that

$$\|\overline{R}^{\varepsilon,n}_{app}(t)\|_{H^s} \le C_s\,\mathcal{E}(t)^{k_0-2},$$

$$\|R^{\varepsilon,n}_{app,*}(t)\|_{H^s} \le C_s\,\varepsilon^{-2}\,\mathcal{E}(t)^{k_0-2}.$$

$$\|\partial_t^\alpha r^{\varepsilon,n}_{app}(t,\cdot)\|_{H^s} \le C'_{s,\alpha}\left(\varepsilon^m\,\mathcal{E}(t)^{k_0+1} + \varepsilon\,\mathcal{E}(t)^{k_0+1}\right).$$

The Fourier transform of V^n and V^{n+1} has compact support in ζ, so that $r^{\varepsilon,n}_{app} = \varepsilon^{m+n}(V^n_* + \varepsilon V^{n+1}_*)$ is such that $r^{\varepsilon,n}_{app,l} = 0$. As in section 2.3, we can then define W^ε_{app} by (2.24), (2.25) and set $V^\varepsilon_{app} = V^{\varepsilon,n}_{app} + W^\varepsilon_{app}$. It solves (2.27), where the remainder R^ε_{app} satisfies, for all $t \in [0, T^\varepsilon]$,

$$\|\overline{R}^\varepsilon_{app}(t)\|_{H^s} + \varepsilon^2\|R^\varepsilon_{app,*}(t)\|_{H^s} \le C_s\,\mathcal{E}(t)^{k_0-2} \tag{3.17}$$

We deduce (as in (2.26))

$$\begin{aligned}\|R^\varepsilon_{app,l}(t)\|_{H^s} &\le \|\overline{R}^\varepsilon_{app,l}(t)\|_{H^s} \;+\; \|R^\varepsilon_{app,*,l}(t)\|_{H^s}\\ &\le \|\overline{R}^\varepsilon_{app}(t)\|_{H^s} \;+\; C\,\varepsilon^2\,\|R^\varepsilon_{app}(t)\|_{H^{s+1/2}}\\ &\le C'_s\,\mathcal{E}(t)^{k_0-2}.\end{aligned}\tag{3.18}$$

<u>Conclusion</u>

We fix $s \ge 5$, $\eta > 0$ to be chosen later. With notations of proposition 2.1, equation (2.15), we choose $k_0 > \max(4, 2 + \lambda^{-1}\mathcal{C}_s)$. Let V^ε the solution of (2.2), with initial data $V^\varepsilon|_{t=0} = V^\varepsilon_{app}|_{t=0}$. Let $W^\varepsilon = V^\varepsilon - V^\varepsilon_{app}$, and

$$T(\varepsilon) = \sup\left\{t \in [0, T^\varepsilon], \quad \forall u \in [0,t],\ \alpha_s(W^\varepsilon; u) \le 1/2\right\}.$$

We apply proposition 2.1 to W^ε. It yields: for all $t \in [0, T(\varepsilon))$,

$$\alpha_s(W^\varepsilon;t) \le 2\mathcal{C}_s\int_0^t \alpha_s(W^\varepsilon;u)du + \frac{4}{3}\mathcal{C}_s\Big(\varepsilon^6\int_0^t \|R^\varepsilon_{app,h}(u)\|^2_{H^s}du + \int_0^t \|R^\varepsilon_{app,l}(u)\|^2_{H^s}du\Big).$$

We deduce from (3.18)

$$\alpha_s(W^\varepsilon;t) \le 2\mathcal{C}_s\int_0^t \alpha_s(W^\varepsilon;u)du \;+\; C_s\,\mathcal{E}(t)^{2(k_0-2)}.$$

With our choice for k_0, Gronwall's lemma implies

$$\alpha_s(W^\varepsilon;t) \le\; C_s\,\mathcal{E}(t)^{2(k_0-2)}.$$

For $t^\varepsilon = T^\varepsilon - \sigma$, σ independent of ε, large enough:

$$C_s\,\mathcal{E}(t)^{2(k_0-2)} \;<\; e^{-2(k_0-2)\lambda^0\sigma} < 1/4.$$

This shows that $T(\varepsilon) \ge t^\varepsilon$, and that

$$\|W^\varepsilon_l(t^\varepsilon)\|^2_{H^s} + \varepsilon^2\|W^\varepsilon(t^\varepsilon)\|^2_{H^s} < exp(-2\lambda^0\sigma), \quad \forall s.$$

Remind also that, up to consider a larger σ,

$$\|V^\varepsilon_{app} - \overline{V}^0\|^2_{H^s} \;\le\; e^{-2\lambda^0\sigma}.$$

We can now conclude the proof of theorem 1.5. We introduce

$$\begin{aligned}(v^\varepsilon, b^\varepsilon)^t(t,\mathrm{x}) &= V^\varepsilon\left(t, \mathrm{x}, \varepsilon^{-4}t, \varepsilon^{-2}\mathrm{x}\right)\\ &= (w^\varepsilon, \beta^\varepsilon)\left(t, \mathrm{x}, \varepsilon^{-4}t, \varepsilon^{-2}\mathrm{x}\right),\end{aligned}$$

which is solution of the original system (1.5). Rapid computations lead to

$$\begin{aligned}\|b^\varepsilon(t)\|^2_{L^2_\mathrm{x}} &\ge \|\beta^\varepsilon(t)\|^2_{L^2_{\mathrm{x},\tau,\theta}} \;-\; C\,\varepsilon^2\|\beta^\varepsilon(t)\|^2_{H^1_{\mathrm{x},\tau,\theta}}\\ &\ge \|\overline{\beta}^\varepsilon_l(t)\|^2_{L^2_\mathrm{x}} \;- C\,\varepsilon^2\|\beta^\varepsilon(t)\|^2_{H^1_{\mathrm{x},\tau,\theta}}\\ &\ge \frac{\varepsilon^m}{2}\|\overline{\beta}^0_l(t)\|^2_{L^2_\mathrm{x}} \;-\; C'\Bigg(\|\overline{V}^\varepsilon_{app,l}(t) - \varepsilon^m\overline{V}^0_{app,l}(t)\|^2_{L^2_\mathrm{x}} + \|\overline{W}^\varepsilon_l(t)\|^2_{L^2_\mathrm{x}}\\ &\quad + \varepsilon^2\left(\varepsilon^m\|\overline{\beta}^0(t)\|^2_{H^1_\mathrm{x}} + \|V^\varepsilon_{app}(t) - \varepsilon^m\overline{V}^0_{app}(t)\|^2_{H^1_{\mathrm{x},\tau,\theta}} + \|W^\varepsilon(t)\|^2_{H^1_{\mathrm{x},\tau,\theta}}\right)\Bigg)\end{aligned}$$

where C and C' are positive constants independent of ε and η. As the Fourier transform of $\overline{\beta}^0$ has compact support, we deduce that

$$\overline{\beta}^0 = \overline{\beta}^0_l, \quad \|\overline{\beta}^0\|_{H^1_x} \leq R\,\|\overline{\beta}^0\|_{L^2_x}$$

for some $R > 0$.

Using previous bounds, it yields, for ε small enough:

$$\|b^\varepsilon(t)\|^2_{L^2} \geq C_0\, e^{-\lambda^0\sigma} - C_1\, e^{-2\lambda^0\sigma} \geq \delta > 0,$$

up to consider a larger σ. Theorem 1.5 follows.

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