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Weakly nonlinear analysis of the α effect

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We address mathematical issues raised by the so-called $\boldsymbol{\alpha}$ effect of dynamo theory, which is a dynamo mechanism arising in conducting flows with small scale fluctuations. Analytical results on the $\boldsymbol{\alpha}$ effect concern the linear induction equation, and are usually claimed to hold for the whole magnetohydrodynamics (MHD) system, as long as the amplitude of the perturbations is small. We discuss the justification of that claim, in the case of periodic fluctuations of the fields. We show a nonlinear instability result on the MHD system, that predicts dynamo action for a large class of high frequency periodic flows, up to the fully nonlinear regime.

Keywords: Magnetohydrodynamics, dynamo theory, α-effect, nonlinear instability, singular perturbation.

1 Introduction

Fluid dynamos are mechanisms that generate magnetic energy through the movement of a conducting fluid. In the context of incompressible flows, relevant equations are the so-called magnetohydrodynamics (MHD) equations. In their classical dimensionless form, they read

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \dfrac{1}{\mathrm{Re}}\nabla^2\mathbf{u} = \Lambda(\nabla\times\mathbf{b})\times\mathbf{b} + \mathbf{f}, \\ \partial_t \mathbf{b} - \nabla\times(\mathbf{u}\times\mathbf{b}) - \dfrac{1}{\mathrm{Rm}}\nabla^2\mathbf{b} = 0, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0, \end{cases} \tag{1}$$

where

$$\mathbf{u} = \mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^3, \quad p = p(t,\mathbf{x}) \in \mathbb{R}, \quad \mathbf{b} = \mathbf{b}(t,\mathbf{x}) \in \mathbb{R}^3$$

are dimensionless velocity, pressure and magnetic fields. The parameters Re, Rm and Λ are the hydrodynamic Reynolds number, the magnetic Reynolds number and the Elsasser number respectively. Finally, $\mathbf{f}$ is a dimensionless bulk force, modelling for instance mechanical constraints or convection sources.

Roughly speaking, dynamo theory deals with the stability of solutions

$$(\mathbf{u},\mathbf{b}) \;=\; (\mathbf{u}(t,\mathbf{x}), 0)$$

of system (1). The basic idea is that the self-excited term $\nabla\times(\mathbf{u}\times\mathbf{b})$ may generate exponential growth of the magnetic field, despite the dissipation term $-\mathrm{Rm}^{-1}\nabla^2\mathbf{b}$. It is widely accepted that dynamo action takes place in the Earth, in the Sun and in many other planets and stars. We refer to review papers by Fearn (1998) and Gilbert (2003) for a good overview and extensive list of references. To understand the actual dynamo processes in these astrophysical objects is a tremendous challenge. The main difficulty is that Re and Rm are very high, leading to intricate turbulent flows. Moreover, numerous factors are

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potentially involved in the dynamo, including geometry of the streamlines, separation of scales, convective effects, differential rotation, compressibility, and so on.

In the last fifty years, many works have been devoted to the linearization of system (1), that is to the induction equation

$$\partial_t \mathbf{b} - \nabla \times (\mathbf{u} \times \mathbf{b}) - \frac{1}{\mathrm{Rm}} \nabla^2 \mathbf{b} = 0.$$

The effect of the Laplace force on the velocity field is neglected, and one looks for flows $\mathbf{u}$ that create instabilities in $\mathbf{b}$. Such flows are called *kinematic dynamos*.

Very briefly, *to exhibit large scale flows that are kinematic dynamos is not easy*. First, the velocity fields should be genuinely three-dimensional, as stated by various antidynamo theorems: see for instance the pioneering papers by Cowling (1934), Zeldovich *et al* (1980), as well as refinements by Ivers *et al* (1988a,b), Arnold *et al* (1998). Besides, in the régime of very small magnetic diffusivity, the flow must have a positive Lyapunov exponent to generate instabilities on the advective timescale ("fast dynamos"). This condition, corresponding to exponential stretching of the particles somewhere in the flow , is discussed in the book by Arnold *et al* (1998) and analysed to its full extent in articles by Vishik (1989), Friedlander *et al* (1993, 1995). It is believed to be "generic", that is shared by "almost all" realistic flows. However, up to our knowledge, such claim has not been mathematically stated. Moreover, the necessary condition of exponential stretching is not known to be sufficient. As a result, only a few fast dynamos have been rigorously identified (among which the famous "ABC flows"), often missing physical features, such as realistic boundary conditions. Note that the situation is also quite difficult from the experimental point of view, where to obtain dynamo in an unconstrained MHD flow is still a challenge.

In parallel to the search of large scale dynamos, special attention has been paid to *irregular or small scale flows, exhibiting oscillation or concentration phenomena*. Let us for instance mention the Ponomarenko flow, which is a curved vortex sheet (see Ponomarenko (1973), Gilbert (1988)).

Among the small scale mechanisms that have been identified, one of the most famous is the so-called $\boldsymbol{\alpha}$ effect. It is based on a separation of scales. The velocity and magnetic fields are assumed to vary on time and length scales τ and l, much smaller than the typical macro scales T and L. Introducing the ratios $\lambda = \tau/T \ll 1$ and $\beta = l/L \ll 1$, one can write this with little formalism:

$$\mathbf{u} \approx \tilde{\mathbf{u}}\left(t, \mathbf{x}, \lambda^{-1} t,\, \beta^{-1} \mathbf{x}\right) \;+\; \bar{\mathbf{u}}\left(t, \mathbf{x}\right),$$
$$\mathbf{b} \approx \tilde{\mathbf{b}}\left(t, \mathbf{x}, \lambda^{-1} t,\, \beta^{-1} \mathbf{x}\right) \;+\; \bar{\mathbf{b}}\left(t, \mathbf{x}\right),$$

where $\tilde{\mathbf{u}}$ (resp. $\tilde{\mathbf{b}}$) is the fluctuating part of the field, and $\bar{\mathbf{u}}$ (resp. $\bar{\mathbf{b}}$) is its mean part. The basic idea is that the "average" of the fluctuating term $\nabla \times (\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{b}}})$ can have a destabilizing effect on the mean field $\bar{\mathbf{b}}$. If we consider only the induction equation, $\mathbf{u}$ being given, then $\tilde{\mathbf{b}}$ is related linearly to $\overline{\mathbf{b}}$, and one can write (formally):

$$\nabla \times (\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{b}}}) \;=\; \nabla \times \left(\boldsymbol{\alpha}\, \overline{\mathbf{b}}\right)$$

for some operator usually denoted $\boldsymbol{\alpha}$. A huge literature has been devoted to the $\boldsymbol{\alpha}$ effect, since the pioneering works of Parker (1955) and Braginsky (1964). Again, we refer to the review paper by Gilbert (2003) for appropriate references. Note that the notion of fluctuations and average can be understood either in a spatial sense or in a probabilistic sense (see on that last topic Steenbeck *et al* (1966)). Note also that the $\boldsymbol{\alpha}$ effect has been the basis of a succesful dynamo experiment, see Stieglitz *et al* (2001).

Our purpose in this paper is to present a rigorous mathematical analysis of this mechanism, in the framework of the nonlinear system (1). We will restrict ourselves to periodic oscillations of the velocity fields. This special case has been considered in several works, starting from the seminal paper by Roberts (1970). In this paper, an analysis of periodic flows is performed, in the linear framework of the induction equation. It is shown that such flows can create exponential instabilities over large spatial scales: the limit "$\boldsymbol{\alpha}$ model" is derived, and conditions of exponential growth are provided. More specific examples of periodic dynamos are given in Childress (1970), Roberts (1972). The results of G.O. Roberts were partially extended by Vishik in papers Vishik (1986), Vishik (1987), within the setting of two-scale expansions and homogenisation. As in Roberts (1970), an homogenised $\boldsymbol{\alpha}$ model is derived, relying on arguments of perturbation theory for linear operators. Let us also mention Zheligovsky (1991) for a generalization to flows in axisymmetric reservoirs.

Up to our knowledge, all these mathematical studies were limited to the linear setting, with proofs based on arguments of operator theory, applied to the linear induction operator. For the whole nonlinear MHD system (1), such arguments do not work anymore, as nonlinearities introduce error terms in the equations. The general feeling is that as long as the amplitude of the magnetic field is small (say $O(\delta)$ with small δ), the nonlinear terms will be of size δ^2, much smaller than δ and thus negligible. This naive "perturbative" argument is known to hold for finite dimensional ordinary differential equations (ODE's). It is still true (although far more difficult to prove!) for ODE's in infinite dimensions, see the paper Shatah *et al* (2000). *However, no general result is available for partial differential equations (PDE's).* To go from linear to nonlinear instability in a given PDE may be highly nontrivial, far from the intuition of finite dimensional problems. It has generated in the last few years a lot of research, notably connected to hydrodynamics (among many, let us mention Friedlander *et al* (1997), Desjardins *et al* (2003), Friedlander *et al* (2003a,b), Strauss *et al* (2004)). For instance, to extend the linear instability result of Vishik (1989) to nonlinear MHD equations is an open problem, as well as the analogue for incompressible Euler equations, *cf* Friedlander *et al* (1993).

Before describing the nonlinear analysis of the $\boldsymbol{\alpha}$ effect, let us underline its main difficulties. The MHD equations involve quadratic terms $Q(\mathbf{V},\mathbf{V})$, with $\mathbf{V} = (\mathbf{v},\mathbf{b})$ describing the velocity $\mathbf{v}$ and magnetic field $\mathbf{b}$. *Such quadratic terms contain derivatives of* $\mathbf{V}$. Because of these derivatives, denoting X the space where the semigroup of the linearized operator is controlled, one does not have

$$\|Q(\mathbf{V},\mathbf{V})\|_X \;\le\; C\|\mathbf{V}\|_X^2.$$

where $\|\;\|_X$ is a norm on X (classically, X is the energy space L^2, or a Sobolev space H^s). This is the crucial difference between ODE's and PDE's with respect to instability issues, and explains the failure of a naive perturbative argument. Hence, errors due to nonlinear terms have to be carefully estimated, in an appropriate norm. Moreover, in the case of the $\boldsymbol{\alpha}$ effect, spatial oscillations with high frequency ε^{-1} create "singular" terms of high amplitude ε^{-1} in the equations. Thus, to find a norm that controls both nonlinearity and singularities is a main problem. As will be detailed below, we are able to solve it in the case of periodic oscillations, and prove a nonlinear instability result. This mathematical result predicts that dynamo action occurs for a large class of high frequency velocity fields.

We stress that this note only gives the main lines of the reasoning, and *is intended to people without a strong background in mathematical analysis.* For those interested in further details, we refer to Gérard-Varet (2005), where complete and extended mathematical treatment is achieved.

2 Modelling and main result

We wish to consider the dynamo properties of a class of high frequency periodic flows. Namely, we investigate the stability of the solutions $(\mathbf{u}^\varepsilon, 0)$ of (1), where

$$\mathbf{u}^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\,\mathbf{U}\left(\frac{\mathbf{x}}{\varepsilon}\right), \quad \mathbf{U} = \mathbf{U}(\boldsymbol{\theta}), \tag{2}$$

and $\varepsilon \ll 1$ is a small parameter. Note that these velocity fields are steady, with large frequency ε^{-1}, and large amplitude $\varepsilon^{-1/2}$. Note also that assuming $(\mathbf{u}^\varepsilon, 0)$ is a solution of (1) characterizes the source term f. The fields $\mathbf{U}$ are assumed to be smooth, 1-periodic in each direction, and divergence-free. Moreover, one can suppose that

$$\int_{(0,1)^3} \mathbf{U}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = 0,$$

as the average of $\mathbf{U}$ would only be responsible for the advection of the fluid particles. We denote by $\mathcal{F}$ the set of such fields $\mathbf{U}$.

We will consider the stability properties of (2) in the singular régime where ε gets small. The perturbation $(\mathbf{v}^\varepsilon, \mathbf{b}^\varepsilon) = (\mathbf{u} - \mathbf{u}^\varepsilon, \mathbf{b}^\varepsilon)$ of $(\mathbf{u}^\varepsilon, 0)$ then satisfies

$$\begin{cases} \partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon + \nabla q^\varepsilon - \dfrac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}^\varepsilon = \Lambda(\nabla \times \mathbf{b}^\varepsilon) \times \mathbf{b}^\varepsilon, \\ \partial_t \mathbf{b}^\varepsilon - \nabla \times (\mathbf{u}^\varepsilon \times \mathbf{b}^\varepsilon) - \nabla \times (\mathbf{v}^\varepsilon \times \mathbf{b}^\varepsilon) - \dfrac{1}{\mathrm{Rm}} \nabla^2 \mathbf{b}^\varepsilon = 0, \\ \nabla \cdot \mathbf{v}^\varepsilon = \nabla \cdot \mathbf{b}^\varepsilon = 0. \end{cases} \tag{3}$$

We have the following instability result, that we state as a

THEOREM 2.1 *There exists a set $\mathcal{DF}$, open and dense in the set $\mathcal{F}$, such that all $\mathbf{U} \in \mathcal{DF}$ is nonlinearly unstable in the following sense: for all $m \geq 1$, one can find $\delta > 0$, times $t^\varepsilon = 0(|\ln \varepsilon|)$ and perturbations $(\mathbf{v}^\varepsilon, \mathbf{b}^\varepsilon)$ solutions of (3), satisfying:*

$$\int_{\mathbb{R}^3} \left|\mathbf{v}^\varepsilon_{|t=0}\right|^2 + \left|\mathbf{b}^\varepsilon_{|t=0}\right|^2 \leq \varepsilon^m$$

and

$$\int_{\mathbb{R}^3} \left|\mathbf{b}^\varepsilon_{|t=t^\varepsilon}\right|^2 \geq \delta.$$

Broadly speaking, *this theorem tells us that the $\boldsymbol{\alpha}$ effect occurs generically.* By generically, we mean in the topological sense: the set of *dynamo fields* $\mathcal{DF}$ is dense in $\mathcal{F}$. Dynamo action is expressed by the exponential growth of the magnetic field $\mathbf{b}^\varepsilon$: its energy increases from $O(\varepsilon^m)$ to $O(1)$ over a period $O(|\ln \varepsilon|)$, that is exponentially with growthrate $O(1)$. Note that the instability is stated in the natural energy norm, whereas many nonlinear results involve Sobolev norms, that are more mathematically oriented (like in Friedlander *et al* (1997)). Note also that we are not able to follow $\mathbf{b}^\varepsilon$ after t^ε, *i.e.* when the quadratic term becomes comparable in size to the linear terms. In this regard, it is a weakly nonlinear instability result.

3 The instability mechanism

The instability mechanism is of a linear nature, and has been described at the level of the induction equation in Roberts (1970), using tools of perturbation theory. Unfortunately, this approach does not apply to the complete system (3), and we use a Wentzel-Kramers-Brillouin (WKB) two-scale description. We refer to Vishik (1986, 1987), Gilbert (2003) for treatment in the same spirit. To lighten the notations, we assume that

$$\mathrm{Re} = \mathrm{Rm} = \Lambda = 1$$

but straightforward modifications would allow to handle any value of the parameters. System (3) can then be expressed in the following compact way:

$$\partial_t \mathbf{V}^\varepsilon - \nabla^2 \mathbf{V}^\varepsilon = (-\nabla q^\varepsilon, 0) + Q(\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon) + \frac{1}{2} Q(\mathbf{V}^\varepsilon, \mathbf{V}^\varepsilon), \quad \mathrm{Div}\mathbf{V}^\varepsilon = (0,0) \tag{4}$$

where we define $\mathbf{V}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{b}^\varepsilon)$, $\mathbf{U}^\varepsilon = (\mathbf{u}^\varepsilon, 0)$, and for all $\mathbf{V}$, $\mathbf{V}_*$

$$Q(\mathbf{V}, \mathbf{V}_*) \;=\; \begin{pmatrix} \nabla\cdot(\mathbf{b}\otimes\mathbf{b}_* - \mathbf{v}\otimes\mathbf{v}_*) + \nabla\cdot(\mathbf{b}_*\otimes\mathbf{b} - \mathbf{v}_*\otimes\mathbf{v}) \\ \nabla\times(\mathbf{v}\times\mathbf{b}_*) + \nabla\times(\mathbf{v}_*\times\mathbf{b}) \end{pmatrix}, \qquad \mathrm{Div}\,\mathbf{V} \;=\; (\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{b}).$$

3.1 *WKB analysis*

The first step in the instability analysis is to understand the structure of the perturbations $(\mathbf{v}, \mathbf{b})$ satisfying (3). Due to the oscillating structure of the base flow $\mathbf{u}^\varepsilon$, it is natural to look for approximations of the following type:

$$\left(\mathbf{v}^\varepsilon_{app}, \mathbf{b}^\varepsilon_{app}\right)(t, \mathbf{x}) = \varepsilon^{m/2} \sum_{i=0}^{n} \sqrt{\varepsilon}^{\,i} (\mathbf{v}^i, \mathbf{b}^i)(t, \mathbf{x}, \mathbf{x}/\varepsilon)\,, \tag{5}$$

where m, n are integers, and each profile $\mathbf{V}^i = \mathbf{V}^i(t, \mathbf{x}, \boldsymbol{\theta}) = (\mathbf{v}^i, \mathbf{b}^i)(t, \mathbf{x}, \boldsymbol{\theta})$ is smooth, periodic in $\boldsymbol{\theta}$ and reads

$$\mathbf{V}^i(t, \mathbf{x}, \boldsymbol{\theta}) \;=\; \overline{\mathbf{V}}^i(t, \mathbf{x}) + \tilde{\mathbf{V}}^i(t, \mathbf{x}, \boldsymbol{\theta}), \qquad \int_{(0,1)^3} \tilde{\mathbf{V}}^i = 0.$$

The averages $\overline{\mathbf{V}}^i$'s refer to the large scale flows, whereas the $\tilde{\mathbf{V}}^i$'s describe the high frequency oscillations. Note that $\varepsilon^{m/2}$ scales the initial amplitude of the perturbation. Similarly, we write

$$q_{app} = \varepsilon^{(m-1)/2} \sum_{i=0}^{n} \sqrt{\varepsilon}^{\,i} q^i\,(t, \mathbf{x}, \mathbf{x}/\varepsilon)\,.$$

If we plug such expansion in system (4), we obtain at first order $-\nabla^2_{\boldsymbol{\theta}} \mathbf{V}^0 = 0$, which yields $\tilde{\mathbf{V}}^0 = 0$. The next few orders lead to

$$-\nabla^2_{\boldsymbol{\theta}} \mathbf{V}^1 = \left(\nabla_{\boldsymbol{\theta}}\, p^0, 0\right) \;+\; Q_{\boldsymbol{\theta}}\left(\mathbf{U}, \overline{\mathbf{V}}^0\right), \qquad \mathrm{Div}_{\boldsymbol{\theta}}\, \tilde{\mathbf{V}}^1 = (0,0),$$

$$\left(\partial_t - \nabla^2_{\mathbf{x}}\right) \overline{\mathbf{V}}^0 \;=\; (\nabla_{\mathbf{x}}\, \overline{p}^1, 0) \;+\; \overline{Q_{\mathbf{x}}\left(\mathbf{U}, \tilde{\mathbf{V}}^1\right)}, \qquad \mathrm{Div}_{\mathbf{x}}\, \overline{\mathbf{V}}^0 = (0,0).$$

In the case where $\mathbf{v}^0|_{t=0} = 0$, one can check that the velocity field $\mathbf{v}^0 = \mathbf{v}^0(t, \mathbf{x})$ remains zero for all times, and the previous system further simplifies to

$$-\nabla^2_{\boldsymbol{\theta}}\, \mathbf{b}^1 = \nabla_{\boldsymbol{\theta}} \times \left(\mathbf{U} \times \mathbf{b}^0\right),$$

$$\left(\partial_t - \nabla^2_{\mathbf{x}}\right) \mathbf{b}^0 = \nabla_{\mathbf{x}} \times \left(\overline{\mathbf{U} \times \mathbf{b}^1}\right).$$

Note that as expected, the equation for the large scale magnetic field $\mathbf{b}^0(t, \mathbf{x})$ involves a homogenised term, that comes from the interaction of the oscillations of $\mathbf{U}$ and $\mathbf{b}^1$. We can rewrite this in the form

$$\left(\partial_t - \nabla^2_{\mathbf{x}}\right) \mathbf{b}^0 \;=\; \nabla_{\mathbf{x}} \times (\boldsymbol{\alpha}\, \mathbf{b}^0), \tag{6}$$

where $\boldsymbol{\alpha}$ is simply a 3×3 matrix defined by

$$\forall \mathbf{b} \in \mathbb{R}^3, \quad \boldsymbol{\alpha}\mathbf{b} \;=\; \int_{(0,1)^3} \mathbf{U} \times \left((-\nabla^2_{\boldsymbol{\theta}})^{-1} \nabla_{\boldsymbol{\theta}} \times (\mathbf{U} \times \mathbf{b})\right). \tag{7}$$

Equations for the following profiles $\mathbf{V}^i$ are of course of the same type, with source terms coming from lower order profiles.

3.2 *Spectral analysis*

The $\boldsymbol{\alpha}$ effect relies on the exponential behavior of some solutions of (6). Although the expression of $\boldsymbol{\alpha}$ is not very explicit, it is possible to specify analytically the nature of the spectrum and exhibit a growing perturbation. We start by collecting properties of $\boldsymbol{\alpha}$:

PROPOSITION 3.1 *For all* $\mathbf{U} \in \mathcal{F}$, *the matrix* $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{U})$ *is real symmetric. Moreover, the set*

$$\mathcal{DF} = \{\mathbf{U} \in \mathcal{F}, \quad \boldsymbol{\alpha}(\mathbf{U}) \text{ has non zero simple eigenvalues } \}$$

is dense and open in $\mathcal{F}$.

The proof of this proposition is given in Gérard-Varet (2005). The real symmetry of $\boldsymbol{\alpha}$ is easily seen by expressing the integral in (7) with the Fourier coefficients of $\mathbf{U}$. To show the density of $\mathcal{DF}$, starting from an arbitrary field $\mathbf{U}$ in $\mathcal{F}$, and a neighborhood of $\mathbf{U}$ in $\mathcal{F}$, we must find a field $\mathbf{U}_* \in \mathcal{DF}$ in this neighborhood. Therefore, we first truncate the Fourier expansion of $\mathbf{U}$ and consider

$$\mathbf{U}_n(\boldsymbol{\theta}) = \sum_{|k|\leq n} \mathbf{U}_k \, e^{ik\cdot\boldsymbol{\theta}}.$$

For n large enough, $\mathbf{U}_n$ remains in the neighborhood of $\mathbf{U}$. If this truncation $\mathbf{U}_n$ belongs to $\mathcal{DF}$, we take $\mathbf{U}_* = \mathbf{U}_n$. Otherwise, we consider

$$\mathbf{U}_* \;=\; \mathbf{U}_n + \delta_1 \mathbf{V}^1 + \delta_2 \mathbf{V}^2 + \delta_3 \mathbf{V}^3$$

where V^i's are special ABC flows, namely

$$\mathbf{V}^1(\theta_1,\theta_2,\theta_3) = \Big(\cos((n+1)\theta_2),\, \sin((n+1)\theta_1),\, \cos((n+1)\theta_1) + \sin((n+1)\theta_2)\Big)^t,$$
$$\mathbf{V}^2(\theta_1,\theta_2,\theta_3) = \Big(\sin((n+2)\theta_3),\, \sin((n+2)\theta_1) + \cos((n+2)\theta_3),\, \cos((n+2)\theta_1)\Big)^t,$$
$$\mathbf{V}^3(\theta_1,\theta_2,\theta_3) = \Big(\sin((n+3)\theta_3) + \cos((n+3)\theta_2),\, \cos((n+3)\theta_3),\, \sin((n+3)\theta_2)\Big)^t.$$

A simple calculation yields

$$\boldsymbol{\alpha}(\mathbf{U}_*) = \boldsymbol{\alpha}(\mathbf{U}_n) - \delta_1 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} - \delta_2 \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} - \delta_3 \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

For an appropriate choice of δ_i, $\mathbf{U}_*$ remains in the neighborhood of $\mathbf{U}$ and belongs to $\mathcal{DF}$.

Once these properties known, one may perform a spectral analysis of equation (6). One considers the equation

$$\lambda\, \mathbf{b} \;=\; \nabla_{\mathbf{x}} \times \boldsymbol{\alpha}\mathbf{b}, \quad \mathbf{b} = \mathbf{b}(\mathbf{x}).$$

Taking the Fourier transform $\hat{\mathbf{b}} = \hat{\mathbf{b}}(\boldsymbol{\xi})$ of $\mathbf{b}$, we get

$$\lambda\, \hat{\mathbf{b}} \;=\; i\,\boldsymbol{\xi} \times \boldsymbol{\alpha}\hat{\mathbf{b}} - |\boldsymbol{\xi}|^2 \hat{\mathbf{b}}.$$

We look for an unstable perturbation, thus for an eigenvalue λ of

$$\mathbf{A}_{\boldsymbol{\xi}} \; = \; i \; (\boldsymbol{\xi} \times \;) \, \boldsymbol{\alpha} - |\boldsymbol{\xi}|^2$$

with positive real part. Using that $\boldsymbol{\alpha}$ is real symmetric, this resumes to looking for eigenvalues of

$$\mathbf{A}_{\boldsymbol{\eta}} \; = \; i \, (\boldsymbol{\eta} \times \;) \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} \; - \; |\boldsymbol{\eta}|^2,$$

with

$$\boldsymbol{\alpha} = P^t \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} P, \quad \boldsymbol{\eta} = P^t \boldsymbol{\xi},$$

for some orthogonal matrix P. Working with $\mathbf{A}_{\boldsymbol{\eta}}$, one then shows that for $\mathbf{U} \in \mathcal{DF}$ and $\boldsymbol{\eta}$ small enough, there exists a positive eigenvalue $\lambda = \lambda(\boldsymbol{\eta})$, and that this eigenvalue has a non-degenerate maximum at some $\boldsymbol{\eta}_0$. We refer to Gérard-Varet (2005) for this last part, which is a straightforward extension of remarks made in Roberts (1970).

Back to the original variables, this shows the existence of unstable eigenmodes $\mathbf{b}_{\boldsymbol{\xi}}(\mathbf{x}) = \hat{\mathbf{b}}(\boldsymbol{\xi}) e^{i \boldsymbol{\xi} \cdot \mathbf{x}}$, with positive eigenvalue $\lambda = \lambda(\boldsymbol{\xi})$ having a non-degenerate maximum λ_0 at some $\boldsymbol{\xi}_0$. It is then possible to construct an unstable wavepacket $\mathbf{b}^0 = \mathbf{b}^0(t, \mathbf{x})$ centered at $\boldsymbol{\xi}_0$. This wavepacket $\mathbf{b}^0$ is a smooth function satisfying the equation (6). Moreover, using stationary phase type theorem, one can show that its energy satisfies

$$\int_{\mathbb{R}^3} \left| \mathbf{b}^0 \right|^2 (t, \mathbf{x}) \, \mathrm{d}\mathbf{x} \; \sim \; \frac{C}{1+t} \, e^{2\lambda_0 t} \tag{8}$$

i.e. grows exponentially. Note that this construction of wave packets was not necessary in Roberts (1970), Vishik (1986, 1987) where all functions were periodic in the large scale variable $\mathbf{x}$. The model considered here, with $\mathbf{x}$ in the whole space $\mathbb{R}^3$, corresponds to localized instabilities (vanishing at infinity), and might be considered as slightly more realistic.

4 Weakly nonlinear dynamo

4.1 *Approximate growing perturbation*

On the basis of the previous construction, there exists an approximate solution of (3)

$$\mathbf{V}^{\varepsilon}_{app} \; = \; (\mathbf{v}^{\varepsilon}_{app}, \, \mathbf{b}^{\varepsilon}_{app})$$

of the WKB type (5). On one hand, initially,

$$\int_{\mathbb{R}^3} \left| \mathbf{V}^{\varepsilon}_{app}(t=0) \right|^2 = O(\varepsilon^m).$$

On the other hand, by the estimate (8), the first term $\varepsilon^{m/2} \, \mathbf{V}^0$ of $\mathbf{V}^{\varepsilon}_{app}$ has an energy

$$\mathcal{E}(t) \; = \; \frac{C \, \varepsilon^m}{1+t} e^{2\lambda_0 t}$$

We introduce T^{ε} the time at which $\mathcal{E}(T^{\varepsilon}) = 1$. Note that $T^{\varepsilon} = O(|\ln \varepsilon|)$. The next terms in the expansion of $\mathbf{V}^{\varepsilon}_{app}$ solve the same type of equations as $\varepsilon^{m/2} \, \mathbf{V}^0$, with source terms coming from quadratic interaction of lower order terms. As long as $t \leq T^\varepsilon$, their energy remains smaller than $\mathcal{E}(t)$. Precisely, one has (see Gérard-Varet (2005) for details):

$$\int_{\mathbb{R}^3} \left|\mathbf{V}^\varepsilon_{app}(t)\right|^2 \;\geq\; \int_{\mathbb{R}^3} \left|\mathbf{b}^\varepsilon_{app}(t)\right|^2 \;\geq\; C_1\,\mathcal{E}(t) \;-\; C_2\,\mathcal{E}(t)^2$$

which provides some $\delta > 0$, and some time $t^\varepsilon \leq T^\varepsilon$ satisfying

$$\int_{\mathbb{R}^3} \left|\mathbf{b}^\varepsilon_{app}(t = t^\varepsilon)\right|^2 \;\geq\; \delta.$$

If $(\mathbf{v}^\varepsilon_{app}, \mathbf{b}^\varepsilon_{app})$ was an exact solution of the MHD system (3), this would show the instability theorem 2.1. But as it is only an approximate solution, we still need to exhibit an exact solution $(\mathbf{v}^\varepsilon, \mathbf{b}^\varepsilon)$ close to $(\mathbf{v}^\varepsilon_{app}, \mathbf{b}^\varepsilon_{app})$. In short, this means *we must show some stability of this instability mechanism.* We stress that it is far more than a mathematical subtlety. For instance, one could have instabilities developping on shorter time scales, preventing the $\boldsymbol{\alpha}$ effect to hold. Such instabilities are not ruled out by the perturbation theory used in the linear framework (Roberts (1970), Vishik (1986)). Indeed, these perturbative arguments are local: they only describe the eigenvalues of the perturbed operator that stand in the neighborhood of the eigenvalues of the homogenised one.

4.2 *Stability estimates*

Let $\mathbf{V}^\varepsilon$ the solution of (4) satisfying initially,

$$\mathbf{V}^\varepsilon|_{t=0} \;=\; \mathbf{V}^\varepsilon_{app}|_{t=0},$$

where $\mathbf{V}^\varepsilon_{app}$ is the previous expansion with exponential growth. To show that the real perturbation $\mathbf{V}^\varepsilon$ behaves like the approximate one $\mathbf{V}^\varepsilon_{app}$, we have to obtain estimates on $\mathbf{W}^\varepsilon = \mathbf{V}^\varepsilon - \mathbf{V}^\varepsilon_{app}$, uniformly as ε goes to zero. The function $\mathbf{W}^\varepsilon$ satisfies:

$$\partial_t \mathbf{W}^\varepsilon - \nabla^2 \mathbf{W}^\varepsilon = (-\nabla r^\varepsilon, 0) + Q(\mathbf{U}^\varepsilon_*, \mathbf{W}^\varepsilon) + \frac{1}{2} Q(\mathbf{W}^\varepsilon, \mathbf{W}^\varepsilon) + \mathbf{R}^\varepsilon_{app}, \quad \mathrm{Div}\,\mathbf{W}^\varepsilon = 0.$$

where $\mathbf{U}^\varepsilon_* = \mathbf{U}^\varepsilon + \mathbf{V}^\varepsilon_{app}$, and $\mathbf{R}^\varepsilon_{app}$ is a remainder coming from $\mathbf{V}^\varepsilon_{app}$. Broadly, the idea is to find a norm $\|\cdot\|$ for which

$$\begin{aligned}\|\mathbf{W}^\varepsilon(t)\|^2 \;\leq C\Big(\int_0^t \|\mathbf{W}^\varepsilon(s)\|^2\,\mathrm{d}s + \int_0^t \|\mathbf{R}^\varepsilon_{app}(s)\|^2\,\mathrm{d}s \\ + \;\|\mathbf{W}^\varepsilon(t)\|^4 \;+\; \|\mathbf{W}^\varepsilon(t)\|^2 \int_0^t \|\mathbf{W}^\varepsilon(s)\|^2\,\mathrm{d}s\Big).\end{aligned} \tag{9}$$

This type of bound ensures that no instability occurs on a timescale shorter than t^ε, and that W^ε remains small up to this time. Let us emphasize that another energy estimate was established in Vishik (1986),theorem 4.1, for the linearized equations. Unfortunately, the norms or spaces introduced ($H^{1/2}$, $H^{-3/2}$) are too weak to handle the nonlinear convective terms (which contribute to the two last terms in the r.h.s. of (9)). Another approach is needed, for which complete mathematical analysis has been performed in Gérard-Varet (2005). See also Desjardins *et al* (2003) on similar treatment of boundary layer problems. We just focus here on the choice of an appropriate norm.

First, the norm $\|\cdot\|$ should allow to control the quadratic expression Q. As Q involves gradient terms, $\|f\|$ must include derivatives of the function f. The main point is that the control of these derivatives is not easy, because of the oscillations in the velocity field $\mathbf{U}^\varepsilon_*$. When differentiated, such oscillations create singular terms in powers of ε^{-1}, that are not easily bounded uniformly in ε.

One first step to handle these high oscillations is to make them explicitly appear in the MHD system, by the introduction of the auxiliary variable $\boldsymbol{\theta} = \mathbf{x}/\varepsilon$. In other words, we express $\mathbf{W}^\varepsilon$ as

$$\mathbf{W}^\varepsilon(t,\mathbf{x}) \;=\; \mathbf{W}\left(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\right), \quad \mathbf{W} = \mathbf{W}(t,\mathbf{x},\boldsymbol{\theta}),$$

and work with the extended system,

$$\begin{cases} \partial_t \mathbf{W} - \left(\nabla_{\mathbf{x}} + \varepsilon^{-1}\nabla_{\boldsymbol{\theta}}\right)^2 \mathbf{W} = -\left((\nabla_{\mathbf{x}} + \varepsilon^{-1}\nabla_{\boldsymbol{\theta}})r^\varepsilon, 0\right) + \dfrac{1}{\sqrt{\varepsilon}}\left(Q_{\mathbf{x}} + \varepsilon^{-1}Q_{\boldsymbol{\theta}}\right)(\mathbf{U}_*,\mathbf{W}) \\ \qquad\qquad + \dfrac{1}{2}\left(Q_{\mathbf{x}} + \varepsilon^{-1}Q_{\boldsymbol{\theta}}\right)(\mathbf{W},\mathbf{W}) + \mathbf{R}_{app}, \\ \left(\mathrm{Div}_{\mathbf{x}} + \varepsilon^{-1}\mathrm{Div}_{\boldsymbol{\theta}}\right)\mathbf{W} = 0, \end{cases} \tag{10}$$

where

$$\mathbf{U}_* \;=\; \mathbf{U}_*(t,\mathbf{x},\boldsymbol{\theta}) \;=\; \mathbf{U}(\boldsymbol{\theta}) + \varepsilon^{m/2}\sum \sqrt{\varepsilon}^i\mathbf{V}^i(t,\mathbf{x},\boldsymbol{\theta}).$$

Note that system (10) still contains the parameter ε, so that $\mathbf{W}(t,\mathbf{x},\boldsymbol{\theta})$ depends on ε (which is omitted in the notation to lighten the reading). But the oscillations of $\mathbf{U}_*^\varepsilon$ are now coded by the variable $\boldsymbol{\theta}$ in $\mathbf{U}_*$, so that differentiation does not create more and more singularities. In this context, it is classical to use the so-called Sobolev norms

$$\|\mathbf{W}\|_s \;=\; \left(\sum_{\boldsymbol{\alpha},|\boldsymbol{\alpha}|\le s}\int_{\mathbf{x},\boldsymbol{\theta}} |\partial^{\boldsymbol{\alpha}}\mathbf{W}(\mathbf{x},\boldsymbol{\theta})|^2 \mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\theta}\right)^{1/2}$$

which control the energy of W and its derivatives up to order s. However, such norms are not adapted to control the singular terms in system (10), notably the worst term

$$N^\varepsilon(\mathbf{W}) \;=\; \frac{1}{\sqrt{\varepsilon}}\left(Q_{\mathbf{x}} + \varepsilon^{-1}Q_{\boldsymbol{\theta}}\right)(\mathbf{U}_*,\mathbf{W}).$$

For instance, for $s = 0$, a crude energy estimate would provide the inequality

$$\begin{aligned} \|\mathbf{W}(t)\|_0^2 = \int_{\mathbf{x},\boldsymbol{\theta}} |\mathbf{W}(t)|^2 &\le \int_0^t\int_{\mathbf{x},\boldsymbol{\theta}} |N^\varepsilon(\mathbf{W})(s)|\,|\mathbf{W}(s)|\,\mathrm{d}s \;+\; \int_0^t\int_{\mathbf{x},\boldsymbol{\theta}} |R_{app}(s)|\,|\mathbf{W}(s)|\,\mathrm{d}s \\ &\le \frac{C}{\varepsilon^{3/2}}\int_0^t\int_{\mathbf{x},\boldsymbol{\theta}} |\mathbf{W}(s)|^2\,\mathrm{d}s \;+\; \int_0^t\int_{\mathbf{x},\boldsymbol{\theta}} |\mathbf{R}_{app}(s)|^2\,\mathrm{d}s \end{aligned}$$

far from the bound (9). This shows some of the difficulties associated with the choice of the norm.

To derive the inequality (9), the main idea is *to build a norm that distinguishes between high and low Fourier frequencies*. Indeed, the high frequency part diffuses much, and is therefore well controlled despite the singularity: formally, in equation (10), one has the balance

$$-\varepsilon^{-2}\left(\nabla_{\boldsymbol{\theta}}\mathbf{W}\right)^2 \;\sim\; \varepsilon^{-3/2}\,Q_{\boldsymbol{\theta}}(\mathbf{U}_*,\mathbf{W}) \tag{11}$$

which yields a good estimate on the oscillating part $\tilde{\mathbf{W}}$ in terms of the average $\overline{\mathbf{W}}$. Formally,

$$\tilde{\mathbf{W}} \;\approx\; \left(\sqrt{\varepsilon}\,\overline{\mathbf{W}}\right).$$

The difficulty is then to control the average $\overline{\mathbf{W}}$, for which diffusion does not balance the singular term $\varepsilon^{-1/2}\,\overline{Q_{\mathbf{x}}(\mathbf{U}_*, \mathbf{W})}$. However, $\overline{\mathbf{W}}$ is not involved in this term, namely:

$$\varepsilon^{-1/2}\,\overline{Q_{\mathbf{x}}(\mathbf{U}_*, \mathbf{W})} \;=\; \varepsilon^{-1/2}\,\overline{Q_{\mathbf{x}}(\mathbf{U}, \tilde{\mathbf{W}})} \;\approx\; O\left(\overline{\mathbf{W}}\right)$$

using the bound of the high frequency part. In this way, the singularity can be removed and estimates uniform with respect to ε can be obtained. However, this argument has to be refined, as the formal balance (11) is not completely true. In particular, the diffusion operator $(\nabla_{\mathbf{x}} + \varepsilon^{-1}\nabla\boldsymbol{\theta})^2$ can not be approximated by $(\nabla_{\boldsymbol{\theta}})^2$ for large frequencies of $\mathbf{x}$, of order ε^{-1}. For mathematical justification, it is therefore necessary to introduce a smooth truncation function $\chi = \chi(\boldsymbol{\zeta}, \boldsymbol{\xi})$, defined on the Fourier space, where $\boldsymbol{\zeta} \in \mathbb{R}^3$ is the dual variable of $\mathbf{x}$, and $\boldsymbol{\xi} \in \mathbb{T}^3$ is the dual variable of $\boldsymbol{\theta}$. Precisely,

$$\chi(\boldsymbol{\zeta}, \boldsymbol{\xi}) = 1, \quad \text{for } |\boldsymbol{\zeta} + \boldsymbol{\xi}| \leq \delta,$$
$$\chi(\boldsymbol{\zeta}, \boldsymbol{\xi}) = 0, \quad \text{for } |\boldsymbol{\zeta} + \boldsymbol{\xi}| \geq 2\delta,$$

where δ satisfies $0 < \delta < 1/4$. Then, the low and high-frequency parts $\mathbf{W}_l$ and $\mathbf{W}_h$ are defined thanks to *Fourier multipliers*, namely

$$\mathbf{W}_l = \chi(\varepsilon^2 D_{\mathbf{x}}, D_{\boldsymbol{\theta}})\mathbf{W}, \quad \mathbf{W}_h = \left(1 - \chi(\varepsilon^2 D_{\mathbf{x}}, D_{\boldsymbol{\theta}})\right)\mathbf{W} = \mathbf{W} - \mathbf{W}_l.$$

In this framework, one can make the above formal reasoning rigorous, and obtain an inequality of type (9), replacing $\|\mathbf{W}(t)\|^2$ by the following energy functional:

$$\|\mathbf{W}(t)\|^2 \;:=\; \sup_{t' \leq t}\Bigg(\|\mathbf{W}_l(t')\|_5^2 \;+\; \varepsilon^2\,\|\mathbf{W}_h(t')\|_5^2 \;+\; \int_0^{t'} \|\left(\nabla_{\mathbf{x}} + \varepsilon^{-1}\nabla_{\boldsymbol{\theta}}\right)\mathbf{W}_l(u)\|_5^2\,\mathrm{d}u$$
$$+\;\varepsilon^{-1}\int_0^{t'} \|\mathbf{W}_h(u)\|_5^2 \mathrm{d}u \;+\; \varepsilon\int_0^{t'} \|\left(\nabla_{\mathbf{x}} + \varepsilon^{-1}\nabla_{\boldsymbol{\theta}}\right)\mathbf{W}_h(u)\|_5^2\,\mathrm{d}u\Bigg)$$

where $\|\;\|_5$ is one of the Sobolev norms defined above ($s = 5$). Again, we refer to Gérard-Varet (2005) for all details.

We emphasize that such estimate can be extended to more general singular base flows, up to minor modifications. We can add a periodic dependence in time to the profile $\mathbf{U}$, which was considered in Roberts (1970). We can also add a slow spatial dependence, replacing $\mathbf{U}(\boldsymbol{\theta})$ by $\mathbf{U}(\mathbf{x}, \boldsymbol{\theta})$, as was done in Vishik (1986). However, in this last case, a general spectral analysis of the limit $\boldsymbol{\alpha}$ model seems far more difficult (and was not discussed in Vishik (1986)). Besides, the methodology described in the present paper is quite general, and may be used in various instability studies with singular perturbations. Let us mention recent work by Gérard-Varet *et al* (2007) on Ponomarenko type dynamos.

In conclusion, our work points out some mathematical challenges raised by the nonlinearities of the MHD system, even in the context of small amplitude perturbations. In the case of highly oscillating flows, linear instabilities based on the $\boldsymbol{\alpha}$ effect are shown to be sustained up to the fully nonlinear regime. However, this result is still limited to spatial periodic flows, and much remains to be done in the analysis of the probabilistic (moreover turbulent) setting.

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