Derivation of the Planetary Geostrophic Equations

Didier Bresch, David Gérard-Varet, Emmanuel Grenier

Résumé

Dans ce papier, on justifie mathématiquement le lien entre les équations planétaires géostrophiques (PGE) et les équations de Boussinesq hydrostatiques avec force de Coriolis, plus communément appelées équations primitives (PE). Les équations planétaires géostrophiques s'obtiennent à partir des équations primitives lorsque le nombre de Froude Fr, le nombre de Rossby ε et le nombre de Burger Bu tendent vers 0. Ces nombres sont supposés satisfaire $\mathrm{Fr} = O(\varepsilon^{1/2})$ et $\mathrm{Bu} = O(\varepsilon)$, ce qui correspond à la dynamique d'échelle planétaire thermohaline. L'analyse menée ici n'entre pas dans le cadre des nombreuses études réalisées jusqu'à présent sur les asymptotiques en fluides tournants. Elle met en jeu un opérateur singulier qui n'est pas antisymétrique et empêche les estimations d'énergie classiques. Traiter un tel opérateur nécessite de mettre les équations primitives sous *forme normale* en utilisant de manière adéquate les termes visqueux.

Abstract

In this paper, we justify mathematically the derivation of the planetary geostrophic equations (PGE) from the hydrostatic Boussinesq equations with Coriolis force, usually named the primitive equations (PE). The planetary geostrophic equations, which are a classical model of thermohaline circulation, are obtained from the primitive equations as the Froude number Fr, the Rossby number ε and the Burger number Bu go to 0. These numbers are supposed to satisfy $\mathrm{Fr} = O(\varepsilon^{1/2})$ and $\mathrm{Bu} = O(\varepsilon)$ which is relevant to the thermohaline planetary dynamics. The analysis performed here does not follow the same lines as previous asymptotic studies on rotating fluids. It involves a singular operator which is not skew symmetric, and prevents classical energy estimates. To handle such operator requires to put the primitive equations *under normal form*, together with an appropriate use of the viscous terms.

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Abstract

In this paper, we justify mathematically the derivation of the planetary geostrophic equations (PGE) from the hydrostatic Boussinesq equations with Coriolis force, usually named the primitive equations (PE). The planetary geostrophic equations, which are a classical model of thermohaline circulation, are obtained from the primitive equations as the Froude number Fr, the Rossby number ε and the Burger number Bu go to 0. These numbers are supposed to satisfy $\mathrm{Fr} = O(\varepsilon^{1/2})$ and $\mathrm{Bu} = O(\varepsilon)$ which is relevant to the thermohaline planetary dynamics. The analysis performed here does not follow the same lines as previous asymptotic studies on rotating fluids. It involves a singular operator which is not skew symmetric, and prevents classical energy estimates. To handle such operator requires to put the primitive equations *under normal form*, together with an appropriate use of the viscous terms.

Keywords : Singular perturbations, geophysical flows, planetary geostrophic equations, primitive equations, well prepared data, non skew symmetric singular operator, rotating fluids.

AMS subject classification : 35Q30.

1 Introduction

In this paper, we address the derivation of the planetary geostrophic equations, which are classical in geophysics to describe large-scale oceanic circulation. Before we state our results with more details, let us specify the general framework of the study.

Variations of insolation, rainfalls, heat flux between ocean and atmosphere modify strongly the temperature and salinity of the ocean surface. Density currents are generated, that tend to divide the ocean into layers of distinct dynamical properties ("surface" and "deep" waters).The resulting oceanic circulation, called "thermohaline", plays a major role in the heat transfer from tropical to polar regions. It helps in this way to homogeneize the temperature around the globe. It is of particular importance in Western Europe, as the Gulf Stream has a strong warming effect.

The thermohaline circulation has planetary scale $O(100km)$, and is relatively slow: it involves speeds of order $O(1mm.s^{-1})$ far from the coasts, up to $O(1cm.s^{-1})$ near western boundaries. Hence, to understand such circulation requires to derive a simplified model that filters out meso-scale and high frequency phenomena.

To do so, one starts usually from the so-called *primitive equations*, set in a three-dimensional domain Ω:

$$\begin{cases} \partial_t v + u \cdot \nabla v + f \dfrac{v^\perp}{\varepsilon} + \dfrac{\nabla_x p}{\varepsilon} - \nu_x \Delta_x v - \nu_z \partial_z^2 v = 0, \\ \dfrac{\partial_z p}{\varepsilon} = \dfrac{T}{\mathrm{Fr}^2}, \\ \mathrm{div}_x\, v + \partial_z w = 0, \\ \partial_t T + u \cdot \nabla T - \nu'_x \Delta_x T - \nu'_z \partial_z^2 T + \dfrac{\mathrm{Bu}}{\varepsilon} w = 0, \end{cases} \tag{PE}$$

In this system, $u = (v, w)^t = (v_1, v_2, w)^t$ is the velocity, p the pressure, and T the temperature. Variables $x = (x_1, x_2)^t$ and z stand for the horizontal and vertical coordinates. Function $f = f(x_2) > 0$ is the Coriolis parameter, whereas $\varepsilon > 0$, $\mathrm{Fr} > 0$ and $\mathrm{Bu} > 0$ denote the Rossby, the Froude and the Burger number respectively. Viscous coefficients are ν_x, ν_z, ν'_x, $\nu'_z > 0$. Finally, Q is a heat source term.

System (PE) is the standard model used in oceanography. Its physical origin is discussed in J. Pedlosky [14]. Mathematically, we refer to the work of J.–L. Lions, R. Temam and S. Wang in [10]–[11].

In the case of the thermohaline circulation, parameters ε, Fr and Bu satisfy (see [8]):

$$\varepsilon \sim 10^{-4} \text{ to } 10^{-2}, \quad \mathrm{Fr} \sim 10^{-2} \text{ to } 10^{-1}, \quad \mathrm{Bu} \sim 10^{-3} \text{ to } 10^{-1}.$$

It is therefore physically relevant to consider the following asymptotics: $\varepsilon \ll 1$, $\mathrm{Fr} = O(\varepsilon^{1/2})$, and $\mathrm{Bu} = 0(\varepsilon)$. We assume to simplify that

$$\varepsilon \ll 1, \quad \mathrm{Fr} = \varepsilon^{1/2}, \quad \mathrm{Bu} = \varepsilon. \tag{1.1}$$

In the formal limit $\varepsilon \to 0$, this yields the following *planetary geostrophic equations*:

$$\begin{cases} f v^{\perp} + \nabla_x p = 0, \\ \partial_z p = T, \\ \operatorname{div}_x v + \partial_z w = 0, \\ \partial_t T + u \cdot \nabla T - \nu'_x \Delta_x T - \nu'_z \partial_z^2 T + w = Q, \end{cases} \tag{PGE}$$

System (PGE) was originally derived in 1959, by Welander [19], and Robinson and Stommel [15]. Since then, it has been widely used to model planetary scale ocean dynamics: we refer to [8] for good physical overview. It is notably of great computational interest, as it involves only one evolution equation on the temperature (see for instance [2], [8]). Mathematically, it has also been the matter of many studies. The existence of weak solutions has been shown independently in [3] and [12]. Other results have been obtained on similar systems, *c.f.* [2], [16], [17], [18].

The aim of the present paper is to justify mathematically the derivation of (PGE) from (PE). Such rigorous derivation has been up to now an open problem. The main difficulty relies on the singularities of the equations. Indeed, energy estimates on the primitive equations, with the scaling (1.1), in a domain Ω lead to

$$\begin{aligned} \frac{1}{2}\partial_t \left(\int_\Omega |v|^2 + \int_\Omega |T|^2 \right) + \nu_x \int_\Omega |\nabla_x v|^2 + \nu_z \int_\Omega |\partial_z v|^2 + \nu'_x \int_\Omega |\nabla_x T|^2 \\ + \nu'_z \int_\Omega |\partial_z T|^2 = (\frac{1-\varepsilon}{\varepsilon}) \int_\Omega T w + \int_\Omega Q T + \text{ Boundary terms.} \end{aligned} \tag{1.2}$$

Because of the singular term $\varepsilon^{-1}(1-\varepsilon)\int T^\varepsilon w^\varepsilon$, it is not straightforward to find at least an $L^2\left(0, T;\, L^2(\Omega)\right)$ uniform estimate on (u, T). Therefore, classical energy methods fail to prove convergence. It is strongly different from many geophysical systems, for which singular terms are skew symmetric.

The asymptotic limit of such skew symmetric singular equations with temperature have been studied by J.T. Beale and A. Bourgeois [1], F. Charve [4], J.Y. Chemin [5], I. Gallagher [7], D. Iftimie [9]. In their studies, the complete Boussinesq equations are considered, with a fixed Burger number, constant latitude (namely $\mathrm{Bu} = f = 1$), and a Froude number equal to $F\varepsilon^{1/2}$, $F > 0$. This allows to write the system as

$$\begin{cases} \partial_t U + u \cdot \nabla U - \mathcal{D}U + \varepsilon^{-1} LU = \varepsilon^{-1}(-\nabla\Phi, 0), \\ U = (u, T), \\ \mathrm{div} u = 0, \\ U|_{t=0} = U_0, \end{cases} \tag{1.3}$$

where $U = (u, T)^t$, $\mathcal{D}U$ is the viscous term, and

$$L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -F \\ 0 & 0 & F & 0 \end{pmatrix}.$$

In our system, the Burgers number is chosen equal to the Rossby number, so that we can rewrite the system in the following form

$$\begin{cases} \partial_t U + u \cdot \nabla U - DU + we_4 + \varepsilon^{-1} LU + \varepsilon^{-1} NU = \varepsilon^{-1}(-\nabla\Phi, 0), \\ U = (v, 0, T), \\ \mathrm{div} u = 0, \\ U|_{t=0} = U_0, \end{cases} \tag{1.4}$$

where $e_4 = (0, 0, 0, 1)^t$,

$$L = \begin{pmatrix} 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the singular part of our system contains an additional nilpotent term. To handle it, we will use a normal form of system (PE), combined with appropriate use of the viscous terms. Our work is in the spirit of the recent paper [6] on the stability of oscillations in real vanishing viscosity limit. All details will be given in further sections.

The outline of the paper reads as follows. In Section 2, we state our main results. In section 3, we derive formally approximate solutions of system (PE). In section 4, we justify the well-posedness of these approximate solutions. In section 5, we study their linear stability. In section 6, we take care of the nonlinear term.

2 Statement of the results

We now turn to the precise statement of our results. In order to lighten notations, we assume throughout the sequel that horizontal and vertical diffusion are equal: we set

$$\nu := \nu_x = \nu_z, \quad \nu' := \nu'_x = \nu'_z.$$

We also neglect the variation of the Coriolis parameter with the latitude, and suppose to simplify that

$$f \equiv 1. \tag{2.5}$$

This assumption is discussed in remark 2.5 below.

We want to focus on the singular terms of (PE). In particular, we want to avoid the boundary layer problems associated to the borders of bounded domains. Therefore, *we restrict ourselves to periodic solutions*: $(x, z) \in \mathbb{T}^3$. Finally, (PE) becomes

$$\begin{cases} \partial_t v + u \cdot \nabla v + \dfrac{v^\perp}{\varepsilon} + \dfrac{\nabla_x p}{\varepsilon} - \nu \Delta v = 0, \\ \partial_z p = T, \\ \operatorname{div}_x v + \partial_z w = 0, \\ \partial_t T + u \cdot \nabla T - \nu' \Delta T + w = Q. \end{cases} \tag{2.6}$$

and we complete (2.6) with the "well-prepared" initial data

$$v|_{t=0} = 0, \quad T|_{t=0} = 0. \tag{2.7}$$

We now introduce what will be shown to be the limit system of (2.6). It reads

$$\begin{cases} v^\perp + \nabla_x p = 0, \\ \partial_z p = T, \\ \mathrm{div}_x v + \partial_z w = 0, \\ \partial_t T + u \cdot \nabla T - \nu' \Delta T + w = Q, \\ (\partial_t - \nu \Delta_x)\, \Delta_x \overline{p} + \mathrm{curl}_x \left(\overline{v \cdot \nabla_x v} \right) = 0, \end{cases} \tag{2.8}$$

where for any $g = g(x,z)$, we denote

$$\overline{g}(x) = \int_{\mathbb{T}} g(x,z)\, dz, \quad g^*(x,z) = g(x,z) - \overline{g}(x) \tag{2.9}$$

the vertical average and oscillations of g.

Remark 2.1 *System* (2.8) *is not a simple rewriting of equation* (PGE), *which would lead to*

$$\begin{cases} v^\perp + \nabla_x p = 0, \\ \partial_z p = T, \\ div_x v + \partial_z w = 0, \\ \partial_t T + u \cdot \nabla T - \nu' \Delta T + w = Q. \end{cases} \tag{2.10}$$

It involves one more equation, which is due to our periodic boundary conditions. Indeed, (2.10) *is not well-posed: if* (u,T) *is a solution,* $(v + \nabla_x^\perp \varphi, w, T)$ *is still a solution for any* $\varphi = \varphi(t,x)$*. It explains the additional equation on the averaged pressure* $\overline{p}$*.*

System (2.8) has smooth solutions locally in time, as shown by:

Proposition 2.2 *Let* $\tau > 0$ *and* $Q \in \mathcal{C}^\infty\left([0,\tau] \times \mathbb{T}^3\right)$*.*

There exists $0 < \tau_* \le \tau$*, and a unique solution* $(u^0, T^0) \in \mathcal{C}^\infty\left([0,\tau_*] \times \mathbb{T}^3\right)^4$ *of* (2.8)*,* (2.7)*.*

We can now state our main convergence result:

Theorem 2.3 *Let* $\tau > 0$*,* $Q \in \mathcal{C}_c^\infty\left((0,\tau) \times \mathbb{T}^3\right)$*. Let* τ_* *and* (u^0, T^0) *given in proposition 2.2. Let* $s \in \mathbb{N}$*.*

There exists $\varepsilon_0 > 0$*,* $\delta > 0$*, such that for* $\varepsilon \le \varepsilon_0$ *and*

$$\sup_{t,x,z} |\partial_z T^0| < \delta, \tag{2.11}$$

there is a unique solution $(u^\varepsilon, T^\varepsilon) \in \mathcal{C}^\infty\left([0,\tau_*]\times\mathbb{T}^3\right)^4$ *of* (2.6), (2.7). *Moreover,*

$$(u^\varepsilon, T^\varepsilon) \xrightarrow[\varepsilon\to 0]{} (u^0, T^0), \quad \text{in } L^\infty\left(0,\tau_*;\, H^s(\mathbb{T}^3)\right)^4.$$

Remark 2.4 *As will be clear in the proof, all above results also apply when* (x,z) *belongs to* $\mathbb{T}\times\mathbb{R}\times\mathbb{T}$ *or* $\mathbb{R}^2\times\mathbb{T}$. *In particular, the dependence of the heat source* Q *with respect to* x *is not necessarily periodic. This is more reasonable from the physical point of view. For instance, solar heating is clearly decreasing with latitude in northern hemisphere, hence not periodic.*

Remark 2.5 *Assumption 2.5 on the Coriolis parameter* f *is very crude from a physical point of view. In a realistic domain, the* β *plane approximation*

$$f(x_2) = 1 + \beta x_2, \quad 1 + \beta x_2 > 0$$

would be more natural. It is a mathematical simplification that we make for the clarity of exposure. Proposition 2.2 and theorem 2.3 could be extended to x_2-*dependent* f*, up to tedious difficulties.*

Remark 2.6 *Although it can certainly be refined, the smallness assumption 2.11 on* $\sup|\partial_z T^0|$ *is natural. Indeed, as explained in [13], the quantity*

$$1 + \partial_z T^\varepsilon \,\sim\, 1 + \partial_z T^0$$

measures the stratification of the fluid: when $\partial_z T^0$ *is very small, the fluid is well stratified, and so less sensitive to instabilities (at least in the vertical direction). This helps to understand why the smallness assumption plays a role in the convergence result.*

3 Formal derivation

Remind that, for all periodic function g, we denote

$$\overline{g}(x) = \int_0^1 g(x,z)\,dz, \quad g^* = g - \overline{g}.$$

We look for approximate solutions of (2.6) under the form

$$\begin{aligned} u^\varepsilon &= \sum_{i=0}^{\infty} \varepsilon^i u^i, \\ T^\varepsilon &= \sum_{i=0}^{\infty} \varepsilon^i T^i, \\ p^\varepsilon &= \sum_{i=0}^{\infty} \varepsilon^i p^i. \end{aligned} \tag{3.12}$$

We plug Ansatz (3.12) into equations (2.6). At leading order in ε, we recover

$$\begin{cases} v^{0,\perp} + \nabla_x p^0 = 0, \\ \partial_z p^0 = T^0, \\ \operatorname{div}_x v^0 + \partial_z w^0 = 0, \\ \partial_t T^0 + u^0 \cdot \nabla T^0 - \nu' \Delta T^0 + w^0 = Q. \end{cases} \tag{3.13}$$

To close the system, we need to determine an equation on $\overline{p^0}$. The first equation of (3.13) yields

$$\operatorname{div}_x v^0 = \operatorname{div}_x \nabla_x^\perp p^0 = 0.$$

Hence, the divergence-free condition resumes to $\partial_z w^0 = 0$, which is equivalent to

$$(w^0)^* = 0 \tag{3.14}$$

We take terms of order ε^0 in the momentum equations. This leads to

$$\partial_t v^0 + u^0 \cdot \nabla v^0 + (v^1)^\perp + \nabla_x p^1 - \nu \Delta v^0 = 0.$$

Integrating from $z = 0$ to $z = 1$, and using (3.14), we deduce

$$\partial_t \overline{v^0} + \overline{v^1}^\perp + \nabla_x \overline{p^1} - \nu \Delta \overline{v^0} + \overline{v^0 \cdot \nabla_x v^0} = 0. \tag{3.15}$$

From the divergence free condition at order ε^1, we obtain

$$\operatorname{div}_x \overline{v^1} = 0. \tag{3.16}$$

Finally, taking the curl_x of (3.15) and using (3.16), this yields

$$(\partial_t - \nu \Delta_x) \Delta_x \overline{p^0} + \operatorname{curl}_x \left(\overline{v^0 \cdot \nabla_x v^0} \right) = 0, \tag{3.17}$$

where $\overline{v^0} = \nabla_x^\perp \overline{p^0}$. Gathering (3.13) and (3.17), *we thus recover formally system* (2.8).

Proceeding in the same way, one can derive the equations satisfied by higher order terms u^i, p^i, T^i. Namely, for all $i \geq 1$,

$$\begin{cases} v^{i,\perp} + \nabla_x p^i = F^i, \\ \partial_z p^i = T^i, \\ \mathrm{div}_x v^i + \partial_z w^i = 0, \\ \partial_t T^i + u^0 \cdot \nabla T^i + u^i \cdot \nabla T^0 - \nu' \Delta T^i + w^i = G^i, \\ (\partial_t - \nu \Delta_x) \Delta_x \overline{p^i} + \mathrm{curl}_x \left(\overline{v^0 \cdot \nabla_x v^i + v^i \cdot \nabla_x v^0} \right) = H^i, \end{cases} \tag{3.18}$$

where the source terms

$$F^i = -\partial_t v^{i-1} - \sum_{j+k=i-1} u^j \cdot \nabla v^k + \nu \Delta v^{i-1}, \quad G^i = -\sum_{\substack{j+k=i \\ j,k\neq 0}} u^j \cdot \nabla T^k,$$

and

$$H^i = -\mathrm{curl}_x \sum_{\substack{j+k=i \\ j,k\neq 0}} \overline{u^j \cdot \nabla v^k} + \mathrm{curl}_x \left(\overline{\left(\int_0^z \mathrm{curl}_x F^i \right) \partial_z v^0} \right)$$
$$+ (\partial_t - \nu \Delta_x) \,\mathrm{div}_x \overline{F^i}.$$

depend on the lower order profiles. These equations are very close to (2.8), and even simpler as they involve only linear terms. For the sake of brevity, we do not detail their treatment, and focus now on the limit system (2.8).

4 Well-posedness of the limit system

This section is devoted to the proof of proposition 2.2. Readers only interested in the stability study can skip this part at first reading. We first re-express system (2.8) with the vertical average and oscillations of each quantity.

Equation (2.8)a reads

$$v* = \nabla_x^\perp p^*, \quad \overline{v} = \nabla_x^\perp \overline{p}. \tag{4.19}$$

Equation (2.8)b is equivalent to

$$\overline{T} = 0, \quad p^* = \left(\int_0^z T \right)^* . \tag{4.20}$$

Taking into account (4.19), (2.8)c reduces to $w^* = 0$. As $w^* = \overline{T} = 0$, (2.8)d splits into

$$\overline{w} = -\overline{v^* \cdot \nabla_x T} \; + \; \overline{Q},$$
$$\partial_t T + (u \cdot \nabla T)^* - \nu' \Delta T = Q^*.$$

Finally, we notice that

$$\int_{T^2} \overline{v^* \cdot \nabla_x v^*} \, dxdy = \int_{\mathbb{T}^2} \mathrm{div}_x \, \overline{v^* \otimes v^*} = 0,$$
$$\int_{T^2} \overline{v} \cdot \nabla_x \overline{v} \, dxdy = \int_{\mathbb{T}^2} \mathrm{div}_x \, \overline{v} \otimes \overline{v} = 0,$$

so that we can uncurl (2.8)e, and obtain the equivalent Navier-Stokes equations

$$\begin{aligned} &\partial_t \overline{v} + \overline{v} \cdot \nabla_x \overline{v} - \nu \Delta_x \overline{v} + \nabla_x p + \overline{v^* \cdot \nabla_x v^*} = 0, \\ &\mathrm{div}_x \, \overline{v} = 0, \end{aligned}$$

for a pressure term $p = p(t, x)$. Gathering the previous lines, we end with the following formulation of (2.8)

$$\begin{cases} v^* = \nabla_x^\perp \left(\int_0^z T \right)^*, \quad w^* = 0 \\ \overline{w} = -\overline{v^* \cdot \nabla_x T} \; + \; \overline{Q}, \\ \partial_t T + (u \cdot \nabla T)^* - \nu' \Delta T = Q^*, \\ \partial_t \overline{v} + \overline{v} \cdot \nabla_x \overline{v} - \nu \Delta_x \overline{v} + \nabla_x p + \overline{v^* \cdot \nabla_x v^*} = 0, \\ \mathrm{div}_x \, \overline{v} = 0. \end{cases} \tag{4.21}$$

Note that v^*, w^*, and $\overline{w}$ are diagnostic variables.

We state

Proposition 4.1 *Let $s > 3/2 + 1$, $\tau > 0$. Let $Q \in C^0 \left([0, \tau]; \, H^s(\mathbb{T}^3) \right)$. Let $v_0 = \nabla_x^\perp p_0 \in H^s(\mathbb{T}^2)^2$, and $T_0 \in H^s(\mathbb{T}^3)$ with $v_0^* = 0$, $\overline{T_0} = 0$.*
There exists $0 < \tau_* \leq \tau$, and a unique maximal solution (u, T) of (4.21) with

$$\overline{v} \in L^\infty_{loc} \left([0, \tau_*); \, H^s(\mathbb{T}^2) \right)^2 \cap L^2_{loc} \left([0, \tau_*); \, H^{s+1}(\mathbb{T}^2) \right)^2, \tag{4.22}$$

$$T \in L^\infty_{loc}\left([0,\tau_*);\, H^s(\mathbb{T}^3)\right) \cap L^2_{loc}\left([0,\tau_*);\, H^{s+1}(\mathbb{T}^3)\right), \tag{4.23}$$

and

$$\overline{v}|_{t=0} = v_0, \quad T|_{t=0} = T_0. \tag{4.24}$$

Moreover, if $\tau_* < \tau$,

$$\limsup_{t\to\tau_*} \|\overline{v}(t)\|_{L^\infty} + \|T(t)\|_{W^{1,\infty}} = +\infty. \tag{4.25}$$

Proof. We start with a priori energy estimates. For any pair (f,g) of real functions, we shall denote

$$f \lesssim g$$

when

$$f \le C\, g$$

for a constant $C > 0$ (depending only on s, τ, ν, ν', Q and its derivatives). Let (u,T) a solution of (4.21) satisfying (4.22), (4.23), (4.24).

Estimates on v^* *and* $\overline{w}$ *in terms of* T.
As $v^* = \left(\int_0^z \nabla_x^\perp T\right)^*$, we obtain : $\forall t < \tau_*$, $\forall s \ge 0$,

$$\|v^*(t)\|_{L^\infty} \le \|\nabla_x T(t)\|_{L^\infty} \tag{4.26}$$

$$\|v^*(t)\|_{H^s} \le \|\nabla_x T(t)\|_{H^s}. \tag{4.27}$$

We then have

$$\overline{w} = -\overline{v^* \cdot \nabla_x T} \,+\, \overline{Q}$$

so that, using (4.26), we get

$$\|\overline{w}(t)\|_{L^\infty} \le \|\nabla_x T\|^2_{L^\infty} + \|\overline{Q}(t)\|_{L^\infty}. \tag{4.28}$$

Then,

$$\begin{aligned}
\|\overline{w}(t)\|_{H^s} &\le \|v^* \otimes \nabla_x T(t)\|_{H^s} + \|\overline{Q}(t)\|_{H^s} \\
&\le \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\partial^\alpha v^*(t) \otimes \partial^\beta \nabla_x T(t)\|_{L^2} + \|\overline{Q}(t)\|_{H^s} \\
&\le \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\partial^\alpha v^*(t)\|_{L^{2\sigma/|\alpha|}} \|\partial^\beta \nabla_x T(t)\|_{L^{2\sigma/|\beta|}} + \|\overline{Q}(t)\|_{H^s}
\end{aligned}$$

where last line involves Hölder inequality. Using the Gagliardo-Nirenberg inequality : $\forall\ 0 \le |\alpha| \le \sigma$,

$$\|\partial^\alpha g\|_{L^{2\sigma/\alpha}} \lesssim \|g\|_{L^\infty}^{1-|\alpha|/\sigma} \|g\|_{H^\sigma}^{|\alpha|/\sigma}$$

we deduce

$$\begin{aligned}\|\overline{w}(t)\|_{H^s} &\lesssim \|\overline{Q}(t)\|_{H^s} \\ &+ \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|v^*(t)\|_{L^\infty}^{1-|\alpha|/\sigma} \|\nabla_x T(t)\|_{L^\infty}^{1-|\beta|/\sigma} \|v^*(t)\|_{H^\sigma}^{|\alpha|/\sigma} \|\nabla_x T(t)\|_{H^\sigma}^{|\beta|/\sigma},\end{aligned}$$

and thanks to (4.26), (4.27),

$$\|\overline{w}(t)\|_{H^s} \lesssim \|\nabla T(t)\|_{L^\infty} \|\nabla T(t)\|_{H^s} + \|\overline{Q}(t)\|_{H^s} \tag{4.29}$$

Estimates on $\overline{v}$ *and* T. Simple manipulations on (4.21)d lead to

$$\|\overline{v}(t)\|_{H^s}^2 - \|v_0\|_{H^s}^2 + \int_0^t \|\nabla_x \overline{v}\|_{H^s}^2 \lesssim \int_0^t \|\overline{v} \otimes \overline{v}\|_{H^s}^2 + \int_0^t \|v^* \otimes v^*\|_{H^s}^2$$

Proceeding as above, we get

$$\begin{aligned}\|\overline{v} \otimes \overline{v}\|_{H^s} &\lesssim \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\overline{v}\|_{L^\infty} \|\overline{v}\|_{H^\sigma} \\ &\lesssim \|\overline{v}\|_{L^\infty} \|\overline{v}\|_{H^s},\end{aligned}$$

and similarly

$$\|v^* \otimes v^*\|_{H^s} \lesssim \|v^*\|_{L^\infty} \|v^*\|_{H^s},$$

which, with (4.26), (4.27) implies

$$\|v^* \otimes v^*\|_{H^s} \lesssim \|\nabla T\|_{L^\infty} \|\nabla T\|_{H^s}.$$

We obtain the inequality

$$\begin{aligned}\|\overline{v}(t)\|_{H^s}^2 - \|v_0\|_{H^s}^2 + \int_0^t \|\overline{v}(t)\|_{H^s}^2 &\lesssim \int_0^t \|\overline{v}\|_{L^\infty}^2 \|\overline{v}\|_{H^s}^2 \\ &+ \int_0^t \|\nabla T\|_{L^\infty}^2 \|\nabla T\|_{H^s}^2\end{aligned} \tag{4.30}$$

Now, simple manipulations on (4.21)c give

$$\|T(t)\|_{H^s}^2 - \|T_0\|_{H^s}^2 \; + \; \int_0^t \|\nabla T\|_{H^s}^2 \lesssim \int_0^t \|v^* \otimes T\|_{H^s}^2 + \int_0^t \|\overline{v} \otimes T\|_{H^s}^2 + \int_0^t \|\overline{w} T\|_{H^s}^2 + \int_0^t \|Q^*\|_{H^{s-1}}^2.$$

Proceeding as above, we get :

$$\begin{aligned}\|v^* \otimes T\|_{H^s} &\lesssim \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|v^*\|_{L^\infty}^{1-|\alpha|/\sigma} \|T\|_{L^\infty}^{1-|\beta|/\sigma} \|v^*\|_{H^\sigma}^{|\alpha|/\sigma} \|T\|_{H^\sigma}^{|\beta|/\sigma} \\ &\lesssim \|T\|_{W^{1,\infty}} \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\nabla T\|_{H^s}^{|\alpha|/\sigma} \|T\|_{H^s}^{|\beta|/\sigma}\end{aligned}$$

In the same way,

$$\|\overline{v} \otimes T\|_{H^s} \lesssim (\|\overline{v}\|_{L^\infty} + \|T\|_{L^\infty}) \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\overline{v}\|_{H^s}^{|\alpha|/\sigma} \|T\|_{H^s}^{|\beta|/\sigma},$$

and using (4.28), (4.29),

$$\|\overline{w} T\|_{H^s} \; \lesssim \; \|T\|_{W^{1,\infty}}^2 \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \|\nabla T\|_{H^s}^{|\alpha|/\sigma} \|T\|_{H^s}^{|\beta|/\sigma} \; + \|\overline{Q}\|_{H^s} \|T\|_{H^s}.$$

We end up with

$$\begin{aligned}\|T(t)\|_{H^s}^2 - \|T_0\|_{H^s}^2 \; + \; \int_0^t \|\nabla T\|_{H^s}^2 &\lesssim \int_0^t \|Q^*\|_{H^{s-1}}^2 \; + \; \int_0^t \|\overline{Q}\|_{H^s}^2 \|T\|_{H^s}^2 \\ + \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \int_0^t \left(\|T\|_{W^{1,\infty}}^2 + \|T\|_{W^{1,\infty}}^4\right) \|\nabla T\|_{H^s}^{2|\alpha|/\sigma} \|T\|_{H^s}^{2|\beta|/\sigma} \\ + \sum_{\substack{|\alpha|+|\beta|=\sigma \\ 0\le\sigma\le s}} \int_0^t \left(\|\overline{v}\|_{L^\infty}^2 + \|T\|_{L^\infty}^2\right) \|\overline{v}\|_{H^s}^{2|\alpha|/\sigma} \|T\|_{H^s}^{2|\beta|/\sigma}. \quad (4.31)\end{aligned}$$

Thanks to Young's inequality

$$ab \; \le \; \frac{|\alpha|}{\sigma} a^{\sigma/|\alpha|} + \frac{|\beta|}{\sigma} \, b^{\sigma/|\beta|},$$

we have

$$\left(\|T\|^2_{W^{1,\infty}} + \|T\|^4_{W^{1,\infty}}\right) \|\nabla T\|^{2|\alpha|/\sigma}_{H^s} \|T\|^{2|\beta|/\sigma}_{H^s} \le \frac{|\alpha|}{\sigma}\, \delta^{\sigma/|\alpha|} \|\nabla T\|^2_{H^s} + \frac{|\beta|}{\sigma}\, \delta^{-\sigma/|\beta|} \left(\|T\|^2_{W^{1,\infty}} + \|T\|^4_{W^{1,\infty}}\right)^{\sigma/|\beta|} \|T\|^2_{H^s},$$

for all $\delta > 0$, and

$$\left(\|\overline{v}\|^2_{L^\infty} + \|T\|^2_{L^\infty}\right) \|\overline{v}\|^{2|\alpha|/\sigma}_{H^s} \|T\|^{2|\beta|/\sigma}_{H^s} \le \frac{|\alpha|}{\sigma} \|\overline{v}\|^2_{H^s} + \frac{|\beta|}{\sigma} \left(\|\overline{v}\|^2_{L^\infty} + \|T\|^2_{L^\infty}\right)^{\sigma/|\beta|} \|T\|^2_{H^s}.$$

Taking δ small enough, we get finally

$$\|T(t)\|^2_{H^s} - \|T_0\|^2_{H^s} + \int_0^t \|\nabla T\|^2_{H^s} \lesssim \int_0^t \|Q^*\|^2_{H^{s-1}} + \int_0^t \|\overline{Q}\|^2_{H^s}\|T\|^2_{H^s} + \int_0^t \|\overline{v}\|^2_{H^s} + \int_0^t F\left(\|\overline{v}\|_{L^\infty} + \|\mathbb{T}\|_{W^{1,\infty}}\right) \|T\|^2_{H^s}, \quad (4.32)$$

for some increasing function F.

Existence of a solution. Thanks to estimates (4.30), (4.32), one can build strong solutions of (4.21) through classical Galerkin approximation. Let $\mathbb{P}^n$ the Fourier troncature of order n. For $n \ge 1$, we define (u^n, T^n) as the solution of

$$\begin{aligned}
&v^{*,n} = \mathbb{P}^n\, \nabla_x^\perp \left(\int_0^z T^n\right)^*, \quad w^{*,n} = 0 \\
&\overline{w^n} = -\mathbb{P}^n\, \overline{v^{*,n}\cdot\nabla_x T^n} + \mathbb{P}^n\, \overline{Q}, \\
&\partial_t T^n + \mathbb{P}^n\, (u^n\cdot\nabla T^n)^* - \nu'\Delta T^n = \mathbb{P}^n\, Q^*, \\
&\partial_t \overline{v^n} + \mathbb{P}^n\, \overline{v^n}\cdot\nabla_x \overline{v^n} - \nu\Delta_x \overline{v^n} + \nabla_x p^n + \mathbb{P}^n\, \overline{v^{n,*}\cdot\nabla_x v^{n,*}} = 0, \\
&\mathrm{div}_x\, \overline{v^n} = 0.
\end{aligned} \quad (4.33)$$

By standard ODE theory, (u^n, T^n) is well-defined and regular up to a time $0 < \tau^n \le \tau$. Let R large enough. We set

$$t^n = \sup\left\{t \in (0, \tau^n], \quad \|T^n(t)\|^2_{H^s} \le R, \quad \|\overline{v}^n(t)\|^2_{H^s} \le R^3\right\}$$

It is easy to see that (u^n, T^n) satisfy estimates (4.30), (4.32). We deduce from (4.30): $\forall 0 \le t \le t^n$,

$$\|\overline{v^n}(t)\|^2_{H^s} - \|v_0\|^2_{H^s} + \int_0^t \|\nabla_x \overline{v^n}\|^2_{H^s} \lesssim R^3 \int_0^t \|\overline{v^n}\|^2_{H^s} + R\int_0^t \|\nabla T^n\|^2_{H^s}.$$

Thanks to a Gronwall lemma, we thus have: $\exists C > 0$,

$$\|\overline{v^n}(t)\|_{H^s}^2 \lesssim \left(R \int_0^t \|\nabla T^n\|_{H^s}^2 + \|v_0\|_{H^s}^2 \right) \exp(C\, R^3\, t),\ \forall t \leq t^n.$$

In particular, there exists $\tau_1 = \tau_1(R)$ such that: for all $0 \leq t \leq \min(\tau_1, t^n)$,

$$\|\overline{v^n}(t)\|_{H^s}^2 \lesssim R \int_0^t \|\nabla T^n\|_{H^s}^2 + \|v_0\|_{H^s}^2. \tag{4.34}$$

Using this last inequality in (4.32) yields

$$\|T^n(t)\|_{H^s}^2 - \|T_0\|_{H^s}^2 + \int_0^t \|\nabla T^n\|_{H^s}^2 \lesssim \int_0^t \|Q\|_{H^{s-1}}^2 + C \int_0^t \|T^n\|_{H^s}^2$$
$$+ t \left(R \int_0^t \|\nabla T^n\|_{H^s}^2 + \|v_0\|_{H^s}^2 \right) \tag{4.35}$$

where $C = C(R) > 0$ (depends on R, F and Q). Hence, there exists $\tau_2 = \tau_2(R) \leq \tau_1$ such that for all $0 \leq t \leq \min(\tau_2, t^n)$,

$$\|T^n(t)\|_{H^s}^2 + \int_0^t \|T^n(t)\|_{H^s}^2 \lesssim \|T_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + 1 + C \int_0^t \|T^n\|_{H^s}^2. \tag{4.36}$$

Using again Gronwall's lemma,

$$\|T^n(t)\|_{H^s}^2 \lesssim \left(\|T_0\|_{H^s}^2 + |v_0\|_{H^s}^2 + 1 \right) \exp(\tilde{C}\, t)$$

for $\tilde{C} = \tilde{C}(R) > 0$. Hence, one can find $\tau_3 = \tau_3(R) \leq \tau_2$, such that: for all $0 \leq t \leq \min(\tau_3, t^n)$,

$$\|T^n(t)\|_{H^s}^2 \leq R/2 \tag{4.37}$$

Back to (4.36), we obtain

$$\int_0^t \|\nabla T^n\|_{H^s}^2 \lesssim \|T_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + 1 + tCR/2,$$

and back to (4.34),

$$\|\overline{v^n}(t)\|_{H^s}^2 \leq R \left(\|T_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + 1 \right) + tCR^2/2 + \int_0^t \|v^0\|_{H^s}^2.$$

Finally, there exists $\tau_* = \tau_*(R) \leq \tau_4$, such that, for all $0 \leq t \leq \min(\tau_*, t^n)$,

$$\|\overline{v^n}(t)\|_{H^s}^2 \leq R^3/2 \tag{4.38}$$

From (4.37), (4.38), we deduce that $t^n \geq \tau_*$ for all n, and moreover that:

$$\begin{aligned}
&(\overline{v}^n)_n \text{ is bounded in } L^\infty\left(0,\tau_*;\, H^s(\mathbb{T}^2)\right) \cap L^2\left(0,\tau_*;\, H^{s+1}(\mathbb{T}^2)\right)^2, \\
&(T^n)_n \text{ is bounded in } L^\infty\left(0,\tau_*;\, H^s(\mathbb{T}^3)\right) \cap L^2\left(0,\tau_*;\, H^{s+1}(\mathbb{T}^3)\right),
\end{aligned}$$

Using equations (4.33), we deduce

$$\begin{aligned}
&(\partial_t \overline{v}^n)_n \text{ is bounded in } L^2\left(0,\tau_*;\, H^{s-1}(\mathbb{T}^2)\right), \\
&(\partial_t T^n)_n \text{ is bounded in } L^2\left(0,\tau_*;\, H^{s-1}(\mathbb{T}^3)\right).
\end{aligned}$$

Thanks to these bounds, one can extract subsequences and pass easily to the limit in (4.33). This yields a solution (u, T) of (4.21).

Uniqueness and blow-up criterion (4.25). Uniqueness comes from energy estimates on the difference $u = u^1 - u^2$, $T = T^1 - T^2$ between two solutions. The blow-up criterion (4.25) is a straightforward consequence of estimates (4.30), (4.32). As this part is classical and much easier than the previous one, we do not give further details.

As proposition 4.1 is proved, proposition 2.2 follows easily. Indeed, taking $v_0 = 0$, $T_0 = 0$, proposition 4.1 provides a unique solution (u, T) of (2.8), (2.7),

$$(u, T) \in L^\infty\left(0, \tau_{*,s});\, H^s(\mathbb{T}^3)\right)^4$$

for all $s \geq 0$. The blow-up criterion (4.25) ensures that times $\tau_{*,s}$ are bounded from below by a time $\tau_* > 0$. Finally, additional time regularity comes from a bootstrap argument, using the expression of time derivatives given in (2.8). This ends the proof.

5 Linear stability

We now turn to the proof of theorem 2.3. Let $\tau > 0$, and $Q \in \mathcal{C}_c^\infty\left((0,\tau)\times\mathbb{T}^3\right)$.

Thanks to the results of previous sections, one can build accurate approximate solutions of (2.6). Indeed, Proposition 4.1 provides a smooth solution

$$(u^0, T^0) \in \mathcal{C}^\infty\left([0,\tau_*]\times\mathbb{T}^3\right)^4, \quad 0 < \tau_* \leq \tau,$$

of system (2.8), with

$$u^0|_{t=0}, \quad T^0|_{t=0} = 0.$$

Up to minor modifications, one can show inductively that: for all $i \geq 1$, there exists a (unique) smooth solution

$$(u^i, T^i) \in \mathcal{C}^\infty\left([0,\tau_*] \times \mathbb{T}^3\right)^4,$$

of system (3.18), with

$$u^i|_{t=0}, \quad T^i|_{t=0} = 0.$$

If we set

$$u_{app} = \sum \varepsilon^i u^i, \quad T_{app} = \sum \varepsilon^i T^i.$$

the derivation of section 3 shows that (u_{app}, T_{app}) satisfies

$$\begin{cases} \partial_t v_{app} + u_{app} \cdot \nabla v_{app} + \dfrac{v_{app}^\perp}{\varepsilon} + \dfrac{\nabla_x p_{app}}{\varepsilon} - \nu \Delta v_{app} = R_{app,v}, \\ \partial_z p_{app} = T_{app}, \\ \mathrm{div}_x\, v_{app} + \partial_z w_{app} = 0, \\ \partial_t T_{app} + u_{app} \cdot \nabla T_{app} - \nu' \Delta T_{app} + w_{app} = R_{app,T}. \end{cases} \tag{5.39}$$

where the remainder term satisfies: for all $s \geq 0$,

$$\| \,(R_{app,v}, R_{app,T})\, \|_{H^s} = O(\varepsilon^n).$$

To prove theorem 2.3, we will show the stability of this approximate solution, for n large enough. In this section, we focus on the linear stability, and consider the following linearized system : $\forall 0 \leq t \leq \tau_*$,

$$\begin{cases} \partial_t v + u_{app} \cdot \nabla v + u \cdot \nabla u_{app} + \dfrac{v^\perp}{\varepsilon} + \dfrac{\nabla_x p}{\varepsilon} - \nu \Delta v = F_v, \\ \partial_z p = T, \\ \mathrm{div}_x\, v + \partial_z w = 0, \\ \partial_t T + u_{app} \cdot \nabla T + u \cdot \nabla T_{app} - \nu' \Delta T + w = F_T, \\ u|_{t=0} = 0, \quad T|_{t=0} = 0. \end{cases} \tag{5.40}$$

We will prove the following

Theorem 5.1 *For all s, there exists $C > 0$, $\delta > 0$, such that: if*

$$\sup_{0 \leq t \leq \tau^\varepsilon,\, x,\, z} |\partial_z T^0| \, \leq \, \delta, \quad \tau^\varepsilon > 0, \tag{5.41}$$

then, for any smooth functions (u, T) and $F = (F_v, F_\theta)$ satisfying (5.40) on $[0, \tau^\varepsilon]$, the following estimate holds: for all $0 \leq t \leq \tau^\varepsilon$,

$$\varepsilon \|v(t)\|_{H^s}^2 + \varepsilon \int_0^t \|\nabla v\|_{H^s}^2 + \|T(t)\|_{H^s}^2 + \int_0^t \|\nabla T\|_{H^s}^2 \, \leq \, C \int_0^t \|F\|_{H^s}^2 \tag{5.42}$$

As mentioned in the introduction, the singular term of (5.40) is not skew symmetric, and *a priori*, it prevents energy estimates that are uniform in ε. To emphasize this, we set

$$\theta := \frac{T}{\varepsilon^{1/2}}, \quad F_\theta := \frac{F_T}{\varepsilon^{1/2}} \tag{5.43}$$

Hence, system (5.40) turns into

$$\begin{cases} \partial_t v + \dfrac{v^\perp}{\varepsilon} + \dfrac{\nabla_x p}{\varepsilon} + u_{app} \cdot \nabla v + u \cdot \nabla u_{app} - \nu \Delta v = F_v, \\ \dfrac{\partial_z p}{\varepsilon} = \dfrac{\theta}{\varepsilon^{1/2}}, \\ \mathrm{div}_x\, v + \partial_z w = 0, \\ \partial_t \theta + u_{app} \cdot \nabla \theta + \dfrac{u \cdot \nabla T_{app}}{\varepsilon^{1/2}} - \nu' \Delta \theta + \dfrac{w}{\varepsilon^{1/2}} = F_\theta. \end{cases} \tag{5.44}$$

Let $\mathbb{P}$ the Leray projector on bidimensional divergence free vector fields. Equations (5.44)a,b,d read

$$\partial_t V \; + \; \frac{\mathcal{A}^1 V}{\varepsilon} \; + \; \frac{\mathcal{A}^2 V}{\varepsilon^{1/2}} \; + \; \frac{\mathcal{B} V}{\varepsilon^{1/2}} \; + \; \mathcal{C} V \; = \; \mathcal{F} \tag{5.45}$$

with $V := (v, 0, \theta)^t$. The singular terms are

$$\mathcal{A}^1 V = \Big(\mathbb{P}(v^\perp, 0), 0\Big)^t, \quad \mathcal{A}^2 V = \left(\mathbb{P}(0, -\theta),\, w\right)^t,$$

which are skew symmetric, and

$$\mathcal{B} V = \left(0, 0, u \cdot \nabla T_{app}\right)^t.$$

This last term is the "bad" one, as it does not disappear in H^s energy estimates.

The basic idea to prove theorem 5.1 is *to cancel the $O(\varepsilon^{-1/2})$ non skew symmetric term using the $O(\varepsilon^{-1})$ skew symmetric term.* This can be done through a change of variable of the type

$$\tilde{V} := V \; - \; \varepsilon^{1/2} \mathcal{D} V. \tag{5.46}$$

Indeed, (5.46) turns formally (5.45) into:

$$\partial_t \tilde{V} \; + \; \frac{\mathcal{A}^1 \tilde{V}}{\varepsilon} \; + \; \frac{\mathcal{A}^2 \tilde{V}}{\varepsilon^{1/2}} \; + \; \frac{\left[\mathcal{A}^1; \mathcal{D}\right] \tilde{V}}{\varepsilon^{1/2}} \; + \; \frac{\mathcal{B} \tilde{V}}{\varepsilon^{1/2}} \; + \; \tilde{\mathcal{C}} \tilde{V} \; = \; \tilde{\mathcal{F}} \tag{5.47}$$

Broadly speaking, we shall find an operator $\mathcal{D}$ satisfying $\left[\mathcal{A}^1; \mathcal{D}\right] = -\mathcal{B}$, so that

$$\partial_t \tilde{V} + \frac{\mathcal{A}^1 \tilde{V}}{\varepsilon} + \frac{\mathcal{A}^2 \tilde{V}}{\varepsilon^{1/2}} + \tilde{\mathcal{C}} \tilde{V} = \tilde{\mathcal{F}} \tag{5.48}$$

This formulation allows uniform energy estimates. With appropriate use of the viscous term, it shall be enough to conclude to the stability. We call it a *normal form* of the primitive equations, following a terminology encountered in nonlinear geometric optics.

Such idea is inspired by recent work of C. Cheverry [6], in which a class of quasilinear hyperbolic systems with vanishing viscosity is studied. These systems arise in various contexts, notably related to strongly nonlinear geometric optics. The author shows a stability result, using changes of variables of the type (5.46). However, the analysis of [6] does not apply to our problem, as the equations do not share at all the same structure. In particular, the assumptions on the skew symmetric operators of [6] are not fulfilled in our case. Moreover, we point out that this basic idea must be much refined, as will be clear from the proof of theorem 5.1. This proof is divided in three steps:

1. We express the whole velocity u as a function of v^*, θ, and F.

2. We introduce the change of variable of type (5.46). We give the exact expression of operator $\mathcal{D}$, which thanks to the previous step acts only on v^*.

3. We control that the new system (5.48) satisfies a good energy estimate. Theorem 5.1 follows.

5.1 Expression of u with respect to v^*, θ, F.

Let $R_{app} := \sum_{i=1}^{n} \varepsilon^{i-1} T^i$, so that $T_{app} = T^0 + \varepsilon R_{app}$. We rewrite (5.44) with the vertical mean and oscillations of each variable. We apply $\mathbb{P}$ to the equation on $\overline{v}$, to obtain the following equations on the velocity and the temperature:

$$\begin{cases} \partial_t \overline{v} + \mathbb{P}(\overline{u_{\mathrm{app}} \cdot \nabla v + u \cdot \nabla v_{\mathrm{app}}}) - \nu \Delta_x \overline{v} = \overline{F_v}, \\ \partial_t v^* + \dfrac{(v^*)^\perp}{\varepsilon} + \dfrac{\nabla_x (\int_0^z \theta)^*}{\varepsilon^{1/2}} + (u_{\mathrm{app}} \cdot \nabla v + u \cdot \nabla v_{\mathrm{app}})^* - \nu \Delta v^* = (F_v)^*, \\ \mathrm{div}_x \overline{v} = 0, \qquad \mathrm{div}_x v^* + \partial_z w^* = 0, \end{cases} \tag{5.49}$$

and

$$\begin{cases} \overline{\theta} = 0, \\ \partial_t \theta + \dfrac{u \cdot \nabla T^0}{\varepsilon^{1/2}} + \varepsilon^{1/2} u \cdot \nabla R_{\mathrm{app}} + u_{\mathrm{app}} \cdot \nabla \theta - \nu' \Delta \theta + \dfrac{w}{\varepsilon^{1/2}} = F_\theta. \end{cases} \tag{5.50}$$

Note that there is no singular term in (5.49)a. Indeed, as $\overline{v}$ has zero mean and satisfies $div_x \overline{v} = 0$, we can write $\overline{v} = -\nabla_{\mathrm{x}}^{\perp} \phi$ for a smooth potential ϕ. Hence, $\varepsilon^{-1} \overline{v}^{\perp} = \nabla_{\mathrm{x}} (\varepsilon^{-1} \phi)$ is a gradient term, that disappears by applying $\mathbb{P}$. Note also that p^* has been replaced by its expression in terms of θ in (5.49)b. As in previous section, we shall denote

$$f \lesssim g$$

when

$$f \le C g$$

for a constant $C > 0$ (depending only on s, τ, ν, ν', u_{app}, T_{app} and their derivatives), *uniformly bounded as* ε *and* $\sup \partial_z |T^0|$ *go to zero.*

Expression of w. Using the divergence free condition, we get

$$w^* = -\left(\int_0^z \operatorname{div}_x v^* \right)^* . \tag{5.51}$$

As $\overline{\theta} = \overline{T^0} = \overline{R_{app}} = 0$, the equation on the temperature provides

$$\overline{w} = -\overline{v^* \cdot \nabla_x T^0} - \overline{w^* \partial_z T^0} - \varepsilon \overline{v^* \cdot \nabla_x R_{app}} - \varepsilon \overline{w^* \partial_z R_{app}} \\ - \varepsilon^{1/2} \overline{u_{app} \cdot \nabla \theta} + \varepsilon^{1/2} \overline{F_\theta}. \tag{5.52}$$

We define the operators $\mathcal{W}$ and $\mathcal{W}_r$ by:

$$\mathcal{W} f := -\left(\int_0^z \operatorname{div}_x f \right)^* - \overline{f \cdot \nabla_x T^0} - \overline{\left(\int_0^z \operatorname{div}_x f \right)^* \partial_z T^0}, \tag{5.53}$$

$$\mathcal{W}_r f := -\overline{f \cdot \nabla_x R_{app}} - \overline{\left(\int_0^z \operatorname{div}_x f \right)^* \partial_z R_{app}}. \tag{5.54}$$

From (5.51), (5.52), we deduce that

$$w = \mathcal{W} v^* + \varepsilon \mathcal{W}_r v^* - \varepsilon^{1/2} \overline{u_{app} \cdot \nabla \theta} + \varepsilon^{1/2} \overline{F_\theta}. \tag{5.55}$$

The following lemma is straightforward and left to the reader:

Lemma 5.2 *For all $m \in \mathbb{N}$, there exists $C(m) > 0$, such that for all smooth functions f:*

$$\|\mathcal{W} f\|_{H^m} \ + \ \|\mathcal{W}_r f\|_{H^m} \ \lesssim \ C(m) \ (\|\nabla_x f\|_{H^m} + \|f\|_{H^m}) \qquad (5.56)$$

$$\| \left[\partial, \mathcal{W}\right] f\|_{H^m} \ + \ \| \left[\partial, \mathcal{W}_r\right] f\|_{H^m} \ \lesssim \ C(m) \ (\|\nabla_x f\|_{H^m} + \|f\|_{H^m}) \qquad (5.57)$$

with $\partial := \partial_x$, ∂_z or ∂_t.

Expression of $\overline{v}$. We can express $\overline{v}$ in terms of v^*. Indeed, equations (5.49)a, (5.49)c yield

$$\begin{aligned} &\partial_t \overline{v} + \mathbb{P}\left(\overline{v_{app}} \cdot \nabla_x \overline{v} + \overline{v} \cdot \nabla_x \overline{v_{app}}\right) - \nu \Delta_x \overline{v} \\ &= -\mathbb{P}\overline{(u^* \cdot \nabla v_{app} + u_{app} \cdot \nabla v^*)} \ + \ \overline{F_v}, \\ &\mathrm{div}_x \overline{v} = 0. \end{aligned} \qquad (5.58)$$

By integration by parts, we get

$$\overline{w^* \partial_z v_{app}} = \overline{\mathrm{div}_x v^* \, v_{app}}, \qquad \overline{w_{app} \partial_z v^*} = \overline{\mathrm{div}_x v_{app} \, v^*},$$

so that (5.58) can be written

$$\begin{aligned} \partial_t \overline{v} + \mathbb{P}\left(\overline{v_{app}} \cdot \nabla_x \overline{v} + \overline{v} \cdot \nabla_x \overline{v_{app}}\right) - \nu \Delta_x \overline{v} & \\ = \mathcal{L} v^* \ + \ \overline{F_v}, \quad \mathrm{div}_x \overline{v} = 0, & \end{aligned} \qquad (5.59)$$

where

$$\mathcal{L} f := -\mathbb{P}\, \overline{\left(f \cdot \nabla_x v_{app} + \mathrm{div}_x f \, v_{app} + v_{app} \cdot \nabla_x f + \mathrm{div}_x v_{app} \, f\right)} \qquad (5.60)$$

We then introduce the operator $\mathcal{V}$ defined by: for all g, $\chi = \mathcal{V} g$ is the solution of

$$\begin{cases} \partial_t \chi + \mathbb{P}\Big(\overline{v_{\mathrm{app}}} \cdot \nabla_x \chi + \chi \cdot \nabla_x \overline{v_{\mathrm{app}}}\Big) - \Delta_x \chi = g, \\ \mathrm{div}_x \chi = 0, \qquad \chi|_{t=0} = 0. \end{cases} \qquad (5.61)$$

We finally write

$$v = v^* \ + \ \mathcal{V} \mathcal{L} v^* \ + \ \mathcal{V} \overline{F_v}. \qquad (5.62)$$

Lemma 5.3 *For all m, there exists $C(m) > 0$, such that for all smooth functions f and g with $g|_{t=0} = 0$:*

$$\|\mathcal{L} f\|_{H^m} \ \le \ C(m) \ (\|\nabla_x f\|_{H^m} + \|f\|_{H^m}) \qquad (5.63)$$

$$\| \left[\partial, \mathcal{L}\right] f\|_{H^m} \ \le \ C(m) \ (\|\nabla_x f\|_{H^m} + \|f\|_{H^m}), \qquad (5.64)$$

and

$$\|\mathcal{V}g(t)\|_{H^m}^2 \,+\, \int_0^t \|\nabla_x \mathcal{V}g\|_{H^m}^2 \,\lesssim\, C(m) \int_0^t \|g\|_{H^m}^2, \tag{5.65}$$

$$\|\,[\partial,\mathcal{V}]\,g(t)\|_{H^m}^2 \,+\, \int_0^t \|\nabla_x\,[\partial,\mathcal{V}]\,g\|_{H^m}^2 \,\lesssim\, C(m) \int_0^t \|g\|_{H^m}^2, \tag{5.66}$$

where $\partial = \partial_x$ *or* ∂_t.

Proof. Inequalities on $\mathcal{L}$ are straightforward.
Inequality (5.65) comes from a clasical H^m estimate on system (5.61). To prove the estimate (5.66) on space and time commutators, notice that $\chi_\partial := [\partial,\mathcal{V}]\,g$ satisfies

$$\chi_\partial \,=\, \mathcal{V}g_\partial,$$

with

$$g_\partial \,:=\, -\mathbb{P}\,(\,\partial\,\overline{v_{app}}\cdot\nabla_x\,(\mathcal{V}g) \,+\, (\mathcal{V}g)\cdot\nabla_x\,\partial\,\overline{v_{app}})\,.$$

Inequality (5.66) follows, applying (5.65).

5.2 Change of variable

Using notations of (5.55) and (5.62), we express (5.50) as

$$\begin{aligned}
&\overline{\theta} = 0,\\
&\partial_t\theta \,+\, \frac{(v^* \,+\, \mathcal{V}\mathcal{L}v^*)\cdot\nabla_x T^0 \,+\, \mathcal{W}v^*\partial_z T^0}{\varepsilon^{1/2}} \,-\, \overline{u_{app}\cdot\nabla\theta}\,\partial_z T^0\\
&+\, u_{app}\cdot\nabla\theta \,+\, \varepsilon^{1/2}\left(u\cdot\nabla R_{app} \,+\, \mathcal{W}_r v^*\,\partial_z T^0\right)\\
&-\, \nu'\Delta\theta \,+\, \frac{w}{\varepsilon^{1/2}} = -\frac{\mathcal{V}\overline{F_v}}{\varepsilon^{1/2}}\cdot\nabla_x T^0 \,-\, \overline{F_\theta}\partial_z T^0 \,+\, F_\theta^*.
\end{aligned} \tag{5.67}$$

Before we make our change of variable, we introduce another operator $\mathcal{A}^\varepsilon$ defined by:

$$f \mapsto \mathcal{A}^\varepsilon f, \qquad (\mathcal{A}^\varepsilon f)^\perp - \varepsilon\nu\Delta(\mathcal{A}^\varepsilon f) = f. \tag{5.68}$$

Lemma 5.4 *For all* $m \geq 0$*, for all smooth* f*,*

$$\|\mathcal{A}^\varepsilon f\|_{H^m} \,\leq\, \|f\|_{H^m}, \tag{5.69}$$

$$\varepsilon^{1/2}\,\|\nabla\mathcal{A}^\varepsilon f\|_{H^m} \,\leq\, \nu^{-1/2}\|f\|_{H^m}, \tag{5.70}$$

$$\varepsilon\,\|\Delta\mathcal{A}^\varepsilon f\|_{H^m} \,\leq\, 2\,\nu^{-1}\,\|f\|_{H^m}, \tag{5.71}$$

Proof. For all $|\alpha| \le s$, and $\partial^\alpha := \partial^\alpha_{x,z}$, we have

$$(\partial^\alpha \mathcal{A}^\varepsilon f)^\perp - \varepsilon\nu\Delta\left(\partial^\alpha \mathcal{A}^\varepsilon f\right) = \partial^\alpha f$$

The first estimate is obtained testing the equation against $(\partial^\alpha \mathcal{A}^\varepsilon f)^\perp$. The second estimate is obtained testing the equation against $(\partial^\alpha \mathcal{A}^\varepsilon f)$, as

$$\begin{aligned}\varepsilon\nu\|\nabla\partial^\alpha \mathcal{A}^\varepsilon f\|^2_{L^2} &\le \|\partial^\alpha f\|_{L^2}\,\|\partial^\alpha \mathcal{A}^\varepsilon f\|_{L^2} \\ &\le \|\partial^\alpha f\|^2_{L^2} \text{ by (5.69).}\end{aligned}$$

The third one is directly deduced from the equation.

We can now perform our change of variables, in order that the singular term $\varepsilon^{-1/2}\left(v^*\cdot\nabla_x T^0 \,+\, \mathcal{V}\mathcal{L}v^*\cdot\nabla_x T^0 \,+\, \mathcal{W}v^*\,\partial_z T^0\right)^*$ cancels. We set

$$\theta_{new} \;:=\; \theta \;-\; \varepsilon^{1/2}\,\mathcal{D}v^*,$$

$$\begin{aligned}\mathcal{D}f \;&:=\; \left((\mathcal{A}^\varepsilon f)\cdot\nabla_x T^0\right)^* \,+\, \left((\mathcal{V}\mathcal{L}\mathcal{A}^\varepsilon f)\cdot\nabla_x T^0\right)^* \,+\, \left((\mathcal{W}\mathcal{A}^\varepsilon f)\;\partial_z T^0\right)^* \\ &= \mathcal{T}^0 f \,+\, \mathcal{T}^1 f \,+\, \mathcal{T}^2 f.\end{aligned} \tag{5.72}$$

Note that variables $\overline{v}$ and v^* remain unchanged, as well as equations (5.49)a and (5.49)c. Equation (5.49)b becomes

$$\partial_t v^* + \frac{(v^*)^\perp - \nu\varepsilon\Delta v^*}{\varepsilon} + \frac{\nabla_x(\int_0^z \theta_{new})^*}{\varepsilon^{1/2}} - \nabla_x(\int_0^z \mathcal{D}v^*)^* \\ +\; (u_{\mathrm{app}}\cdot\nabla v + u\cdot\nabla v_{\mathrm{app}})^* \;=\; F_v^*.$$

We apply operator $\varepsilon^{1/2}\mathcal{D}$. This yields

$$\begin{aligned}&\partial_t(\varepsilon^{1/2}\mathcal{D}v^*) \;+\; \frac{\mathcal{D}\left((v^*)^\perp - \nu\varepsilon\Delta v^*\right)}{\varepsilon^{1/2}} \;-\; \mathcal{D}\nabla_x(\int_0^z \theta_{new})^* \\ &\quad+\; \varepsilon^{1/2}\mathcal{D}\nabla_x(\int_0^z \mathcal{D}v^*)^* \;+\; u_{app}\cdot\nabla(\varepsilon^{1/2}\mathcal{D}v^*) \;+\; \varepsilon^{1/2}\left[\mathcal{D},\partial_t\right]v^* \\ &\quad+\; \varepsilon^{1/2}\left(\left[\mathcal{D},u_{app}\cdot\nabla\right]v^*\right)^* \;+\; \varepsilon^{1/2}\mathcal{D}\left(u\cdot\nabla v_{app}\right)^* \;=\; \varepsilon^{1/2}\mathcal{D}F_v^*.\end{aligned} \tag{5.73}$$

Note that, by definition of $\mathcal{A}^\varepsilon$,

$$\mathcal{D}\left((v^*)^\perp - \nu\varepsilon\Delta v^*\right) = \left(v^*\cdot\nabla_x T^0 \,+\, \mathcal{V}\mathcal{L}v^*\cdot\nabla_x T^0 \,+\, \mathcal{W}v^*\,\partial_z T^0\right)^*$$

Remark also that $\overline{\theta_{new}} = 0$. Thus, if we substract (5.73) to (5.67)b, and take the oscillatory part of the equation, we obtain:

$$\partial_t \theta_{new} - \overline{u_{app} \cdot \nabla \theta_{new}}\, \partial_z T^0 + (u_{app} \cdot \nabla \theta_{new})^* \\ - \mathcal{D} \nabla_x (\int_0^z \theta_{new})^* - \nu' \Delta \theta_{new} + \frac{w^*}{\varepsilon^{1/2}} = F_{new} + \varepsilon^{1/2} F_u,$$

where

$$F_{new} := -\frac{\mathcal{V} \overline{F_v}}{\varepsilon^{1/2}} \cdot \nabla_x T^0 - \overline{F_\theta}\, \partial_z T^0 + F_\theta^* - \varepsilon^{1/2} \mathcal{D} F_v^* \tag{5.74}$$

and

$$F_u := +\mathcal{D} \nabla_x (\int_0^z \mathcal{D} v^*)^* + [\mathcal{D}, \partial_t]\, v^* + ([\mathcal{D}, u_{app} \cdot \nabla]\, v^*)^* + \mathcal{D}\, (u \cdot \nabla v_{app})^* \\ + \overline{u_{app} \cdot \nabla \mathcal{D} v^*}\, \partial_z T^0 - (u \cdot \nabla R_{app})^* - (\mathcal{W}_r v^*)\, \partial_z T^0 + \nu' \Delta \mathcal{D} v^*. \tag{5.75}$$

Finally, from (5.52), we recover

$$\overline{w} := -\overline{v^* \cdot \nabla_x T^0} - w^* \partial_z T^0 - \varepsilon \overline{w^*\, \partial_z R_{app}} \\ - \varepsilon^{1/2} \overline{u_{app} \cdot \nabla \theta_{new}} - \varepsilon \overline{u_{app} \cdot \nabla \mathcal{D} v^*} + \varepsilon^{1/2} \overline{F_\theta}. \tag{5.76}$$

5.3 Uniform Energy estimates

We now perform some energy estimates that are uniform with respect to ε. To carry these estimates, it is more convenient to "gather" $\overline{v}$ and v^*, and to reformulate previous equations as

$$\partial_t v + \frac{v^\perp}{\varepsilon} + \frac{\nabla_x q}{\varepsilon} - \frac{\nabla_x (\int_0^z \theta_{new})^*}{\varepsilon^{1/2}} + u_{app} \cdot \nabla v + u \cdot \nabla v_{app} - \nu \Delta v \\ = -\nabla_x (\int_0^z \mathcal{D} v^*)^* + F_v, \\ \mathrm{div}_x v + \partial_z w = 0, \\ \overline{\theta_{new}} = 0, \quad \overline{w} \text{ given by (5.76)}, \\ \partial_t \theta_{new} - \overline{u_{app} \cdot \nabla \theta_{new}}\, \partial_z T^0 + (u_{app} \cdot \nabla \theta_{new})^* \\ - \mathcal{D} \nabla_x (\int_0^z \theta_{new})^* - \nu' \Delta \theta_{new} + \frac{w^*}{\varepsilon^{1/2}} = F_{new} + \varepsilon^{1/2} F_u \tag{5.77}$$

for a pressure term $q = q(t, x)$.

Let $\alpha \in \mathbb{N}^3$, $s \geq |\alpha|$, $\partial^\alpha := \partial^\alpha_{x,z}$, and denote by $(\cdot,\cdot)$ the L^2 scalar product. We obtain

$$\begin{aligned}
&\|\partial^\alpha v(t)\|^2_{L^2} \,+\, \nu \int_0^t \|\nabla \partial^\alpha v\|^2_{L^2} \,+\, \|\partial^\alpha \theta_{new}(t)\|^2_{L^2} \,+\, \nu' \int_0^t \|\nabla \partial^\alpha \theta_{new}\|^2_{L^2} \\
= &- \int_0^t ([u_{app}\cdot\nabla, \partial^\alpha] v, \partial^\alpha v]) \,-\, \int_0^t (\partial^\alpha (u \cdot \nabla v_{app}), \partial^\alpha v) \\
&- \int_0^t \left(\partial^\alpha \nabla_x (\int_0^z \mathcal{D} v^*)^*, \partial^\alpha v \right) + \int_0^t (\partial^\alpha F_v, \partial^\alpha v) \\
&- \int_0^t \left(\partial^\alpha \left(\overline{u_{app}\cdot \nabla \theta_{new}}\, \partial_z T^0 \right), \partial^\alpha \theta_{new} \right) \\
&- \int_0^t ([u_{app}\cdot\nabla, \partial^\alpha] \theta_{new}, \partial^\alpha \theta_{new}) \,-\, \int_0^t \left(\partial^\alpha \mathcal{D} \nabla_x (\int_0^z \theta_{new})^*, \partial^\alpha \theta_{new} \right) \\
&+ \int_0^t (\partial^\alpha F_{new}, \partial^\alpha \theta_{new}) \,+\, \varepsilon^{1/2} \int_0^t (\partial^\alpha F_u, \partial^\alpha \theta_{new}) \,=\, \sum_{j=1}^9 I_j.
\end{aligned}$$

Standard computations yield:

$$|I_1| \,\lesssim\, \int_0^t \|v\|^2_{H^s}, \quad |I_4| \,\lesssim\, \int_0^t \|F_v\|^2_{H^s} + \int_0^t \|v\|^2_{H^s},$$

$$|I_5| \,\lesssim\, \eta \int_0^t \|\nabla \theta_{new}\|^2_{H^s} \,+\, \eta^{-1} \int_0^t \|\theta_{new}\|^2_{H^s}, \quad |I_6| \,\lesssim\, \int_0^t \|\theta_{new}\|^2_{H^s},$$

where $\eta < 1$ will be chosen later.

The other terms involve the operator $\mathcal{D} = \mathcal{T}^0 + \mathcal{T}^1 + \mathcal{T}^2$. Thanks to estimates of lemma 5.2 and 5.3, for all m, there exists $C(m) > 0$ such that: for all f with $\int_{\mathbb{T}^3} f = 0$ (so that Poincaré's inequality applies),

$$\begin{aligned}
&\|\nabla \mathcal{T}^0 f\|_{H^m} \,\lesssim\, C(m)\, \|\nabla f\|_{H^m} \\
&\|\mathcal{T}^1 f\|^2_{H^m} \,+\, \int_0^t \|\nabla \mathcal{T}^1 f\|^2_{H^m} + \varepsilon \|\nabla \mathcal{T}^1 f\|^2_{H^m} \,\lesssim\, C(m) \int_0^t \|\nabla f\|^2_{H^m} \\
&\|\mathcal{T}^2 f\|^2_{H^m} \,+\, \varepsilon \|\nabla \mathcal{T}^2 f\|^2_{H^m} \,\lesssim\, C(m)\, \sup |\partial_z T^0|^2 \, \|\nabla f\|^2_{H^m}
\end{aligned} \tag{5.78}$$

This allows to bound I_3 in the following way:

$$\begin{aligned}|I_3| \;\le\; & \int_0^t \|\nabla_x \mathcal{T}^0 v^*\|_{H^s}\|v\|_{H^s} \;+\; \int_0^t \|\nabla_x \mathcal{T}^1 v^*\|_{H^s}\|v\|_{H^s} \\ & + \int_0^t \|\mathcal{T}^2 v^*\|_{H^s}\|\nabla v\|_{H^s} \\ & \lesssim \eta \int_0^t \|\nabla v\|_{H^s}^2 \;+\; \eta^{-1}\int_0^t \|v\|_{H^s}^2 \;+\; \eta^{-1}\sup|\partial_z T^0|^2 \int_0^t \|\nabla v\|_{H^s}^2.\end{aligned}$$

Similarly, we obtain

$$|I_7| \lesssim \eta \int_0^t \|\nabla\theta_{new}\|_{H^s}^2 + \eta^{-1}\int_0^t \|\theta_{new}\|_{H^s}^2 + \eta^{-1}\sup|\partial_z T^0|^2 \int_0^t \|\nabla\theta_{new}\|_{H^s}^2.$$

A simple Cauchy-Schwartz inequality provides

$$\begin{aligned}|I_2| \;\lesssim\; & \int_0^t \|u\|_{H^s}\|v\|_{H^s} \\ \lesssim\; & \int_0^t \|v\|_{H^s}^2 \;+\; \int_0^t \|w^*\|_{H^s}\,\|v\|_{H^s} \;+\; \int_0^t \|\overline{w}\|_{H^s}\,\|v\|_{H^s}.\end{aligned}$$

As

$$\|w^*\|_{H^s} \le \|\nabla v\|_{H^s}, \tag{5.79}$$

we get

$$|I_2| \;\lesssim\; \eta^{-1}\int_0^t \|v\|_{H^s}^2 \;+\; \eta\int_0^t \|\nabla v\|_{H^s}^2 \;+\; \int_0^t \|\overline{w}\|_{H^s}\,\|v\|_{H^s}.$$

Using (5.78), (5.79) in (5.76), we obtain

$$\|\overline{w}\|_{H^s} \;\lesssim\; \|\nabla v\|_{H^s} \;+\; \varepsilon^{1/2}\left(\|\nabla\theta_{new}\|_{H^s} \;+\; \|\nabla v\|_{H^s} \;+\; \|\overline{F_\theta}\|_{H^s}\right).$$

We end up with

$$|I_2| \;\lesssim\; \eta^{-1}\int_0^t \|v\|_{H^s}^2 \;+\; \eta\int_0^t \|\nabla v\|_{H^s}^2 \;+\; \eta\int_0^t \|\nabla\theta_{new}\|_{H^s}^2 \;+\; \varepsilon\int_0^t \|\overline{F_\theta}\|_{H^s}^2$$

for ε small enough.

To treat I_8, we write

$$\begin{aligned}|I_8| \;\le\; & \frac{1}{2}\int_0^t \|F_{new}\|_{H^s}^2 \;+\; \frac{1}{2}\int_0^t \|\theta_{new}\|_{H^s}^2 \\ \lesssim\; & \varepsilon^{-1}\int_0^t \|\mathcal{V}\overline{F_v}\|_{H^s}^2 \;+\; \int_0^t \|F_\theta\|_{H^s}^2 \;+\; \varepsilon\int_0^t \|\mathcal{D}F_v^*\|_{H^s}^2 \;+\; \int_0^t \|\theta_{new}\|_{H^s}^2 \\ \lesssim\; & \varepsilon^{-1}\int_0^t \|\overline{F_v}\|_{H^s}^2 \;+\; \int_0^t \|F_\theta\|_{H^s}^2 \;+\; \int_0^t \|F_v^*\|_{H^s}^2 \;+\; \int_0^t \|\theta_{new}\|_{H^s}^2\end{aligned}$$

using the bounds on $\mathcal{V}$ and the $\mathcal{T}^i$'s.

The integral I_9 is treated with the same arguments. As it contains plenty of terms, we just give one example. We deal with:

$$\left|\varepsilon^{1/2}\int_0^t \left(\partial^\alpha(\nu'\Delta\mathcal{D}v^*), \partial^\alpha\theta_{new}\right)\right| \lesssim \varepsilon^{1/2}\sum_{i=0}^{2}\int_0^t \|\nabla\mathcal{T}^i v^*\|_{H^s}\,\|\nabla\theta_{new}\|_{H^s}$$
$$\lesssim\ \varepsilon^{1/2}\int_0^t \|\nabla\mathcal{T}^0 v^*\|^2_{H^s}\ +\ \varepsilon^{1/2}\int_0^t \|\nabla\mathcal{T}^1 v^*\|^2_{H^s}\ +\ \varepsilon^{1/2}\int_0^t \|\nabla\theta_{new}\|^2_{H^s}$$
$$+\ \eta^{-1}\varepsilon\int_0^t \|\nabla\mathcal{T}^2 v\|^2_{H^s}\ +\ \eta\int_0^t \|\nabla\theta_{new}\|^2_{H^s}.$$

Thanks to inequalities (5.78), we get

$$\left|\varepsilon^{1/2}\int_0^t \left(\partial^\alpha(\nu'\Delta\mathcal{D}v^*), \partial^\alpha\theta_{new}\right)\right|$$
$$\lesssim\ \varepsilon^{1/2}\int_0^t \|\nabla v^*\|^2_{H^s}\ +\ \varepsilon^{1/2}\int_0^t \|\nabla v^*\|^2_{H^s}\ +\ \varepsilon^{1/2}\int_0^t \|\nabla\theta_{new}\|^2_{H^s}$$
$$+\ \eta^{-1}\sup|\partial_z T^0|^2\int_0^t \|v\|^2_{H^s}\ +\ \eta\int_0^t \|\nabla\theta_{new}\|^2_{H^s}$$
$$\lesssim\ \eta\int_0^t \|\nabla v\|^2_{H^s}\ +\ \eta\int_0^t \|\nabla\theta_{new}\|^2_{H^s}\ +\ \eta^{-1}\sup|\partial_z T^0|^2\int_0^t \|\nabla v\|^2_{H^s}$$

for ε small enough. The other terms can be handled in the same manner, with the various estimates given in lemmas 5.2 and 5.3.

If we gather all previous bounds, and sum over all $|\alpha| \le s$, we end up with

$$\|v\|^2_{H^s}\ +\ \int_0^t \|\nabla v\|^2_{H^s}\ +\ \|\theta_{new}\|^2_{H^s}\ +\ \int_0^t \|\nabla\theta_{new}\|^2_{H^s}$$
$$\lesssim\ \eta\left(\int_0^t \|\nabla v\|^2_{H^s}\ +\ \int_0^t \|\nabla\theta_{new}\|^2_{H^s}\right)\ +\ \eta^{-1}\left(\int_0^t \|v\|^2_{H^s}\ +\ \int_0^t \|\theta_{new}\|^2_{H^s}\right)$$
$$+\,\eta^{-1}\sup|\partial_z T^0|^2\left(\int_0^t \|\nabla v\|^2_{H^s}\ +\ \int_0^t \|\nabla\theta_{new}\|^2_{H^s}\right)$$
$$+\ \varepsilon^{-1}\int_0^t \|\overline{F_v}\|^2_{H^s}\ +\ \int_0^t \|F_\theta\|^2_{H^s}\ +\ \int_0^t \|F_v\|^2_{H^s}.$$

We choose $\eta > 0$ small enough, so that the first term of the right handside is absorbed by the gradient terms at the left. We are left with

$$\|v\|_{H^s}^2 + \int_0^t \|\nabla v\|_{H^s}^2 + \|\theta_{new}\|_{H^s}^2 + \int_0^t \|\nabla \theta_{new}\|_{H^s}^2$$
$$\lesssim \left(\int_0^t \|v\|_{H^s}^2 + \int_0^t \|\theta_{new}\|_{H^s}^2 \right) + \sup |\partial_z T^0|^2 \left(\int_0^t \|v\|_{H^s}^2 + \int_0^t \|\theta_{new}\|_{H^s}^2 \right)$$
$$+ \varepsilon^{-1} \int_0^t \|\overline{F_v}\|_{H^s}^2 + \int_0^t \|F_\theta\|_{H^s}^2 + \int_0^t \|F_v\|_{H^s}^2.$$

Remind that $\theta_{new} = \theta - \varepsilon^{1/2} \mathcal{D} v^*$. Using again estimates (5.78), we obtain

$$\|\theta_{new}(t)\|_{H^s}^2 - \int_0^t \|v\|_{H^s}^2 - \sup |\partial_z T^0|^2 \|v(t)\|_{H^s}^2 \lesssim \|\theta(t)\|_{H^s}^2 \lesssim \|\theta_{new}(t)\|_{H^s} + \int_0^t \|v\|_{H^s}^2 + \sup |\partial_z T^0|^2 \|v(t)\|_{H^s}^2,$$

and

$$\int_0^t \|\nabla \theta\|_{H^s}^2 - \sup |\partial_z T^0|^2 \int_0^t \|v\|_{H^s}^2 \lesssim \int_0^t \|\nabla \theta_{new}\|_{H^s}^2.$$

Hence, for ε small enough, and for $\sup |\partial_z T^0| < \alpha$, α small enough, we have:

$$\|v\|_{H^s}^2 + \int_0^t \|\nabla v\|_{H^s}^2 + \|\theta\|_{H^s}^2 + \int_0^t \|\nabla \theta\|_{H^s}^2$$
$$\lesssim \left(\int_0^t \|v\|_{H^s}^2 + \int_0^t \|\theta\|_{H^s}^2 \right) + + \varepsilon^{-1} \int_0^t \|\overline{F_v}\|_{H^s}^2 + \int_0^t \|F_\theta\|_{H^s}^2 + \int_0^t \|F_v\|_{H^s}^2.$$

Back to original variables v and T, theorem 5.1 follows from a Gronwall's lemma.

6 Nonlinear stability

Thanks to theorem 5.1, we can take care of the nonlinear terms and prove theorem 2.3. Let $\tau > 0$, $Q \in \mathcal{C}_c^\infty \left((0,\tau) \times \mathbb{T}^3 \right)$. Let $n > 2$, and

$$\left(u_{app} = \sum_{i=0}^n \varepsilon^i u^i, \qquad T_{app} = \sum_{i=0}^n \varepsilon^i T^i \right) \in \mathcal{C}^\infty \left([0, \tau_*] \times \mathbb{T}^3 \right)^4$$

the approximate solution built at the beginning of section 5.

Let $s > 3/2 + 1$. At fixed ε, the local existence theory of smooth solutions for (2.6) is classical. There exists a unique maximal solution

$$(u^\varepsilon, T^\varepsilon) \in \mathcal{C}^\infty\left([0, \tau^\varepsilon) \times \mathbb{T}^3\right)^4.$$

Moreover, the lifespan τ^ε satisfies one of the following conditions:

- $\tau^\varepsilon \geq \tau_*$
- $\tau^\varepsilon < \tau_*$, and $\liminf_{t \to \tau^\varepsilon} \|(u^\varepsilon, T^\varepsilon)(t)\|_{H^s} = +\infty$.

We will show that the second possibility does not occur.

Let $u := u^\varepsilon - u_{app}$, $T = T^\varepsilon - T_{app}$. For all $0 \leq t \leq \min(\tau^\varepsilon, \tau_*)$, (u, T) satisfies (5.40) with

$$\begin{aligned} F_v &:= -u \cdot \nabla v \; + \; R_{app,v} \\ F_T &:= -u \cdot \nabla T \; + \; R_{app,T}. \end{aligned}$$

We define

$$\alpha^\varepsilon(t) \; := \; \sup_{0 \leq t' \leq t} \left(\varepsilon \|v(t')\|^2_{H^s} + \|T(t')\|^2_{H^s} + \varepsilon \int_0^t \|\nabla v\|^2_{H^s} + \int_0^t \|\nabla T\|^2_{H^s} \right)$$

We apply theorem 5.1: for all $0 \leq t \leq \min(\tau^\varepsilon, \tau_*)$,

$$\begin{aligned} \alpha^\varepsilon(t) \; &\lesssim \; \int_0^t \|u \cdot \nabla v\|^2_{H^s} \; + \; \int_0^t \|u \cdot \nabla T\|^2_{H^s} \; + \; \int_0^t \|R_{app}\|^2_{H^s} \\ &\lesssim \; \int_0^t \|v \cdot \nabla_x v\|^2_{H^s} \; + \; \int_0^t \|v \cdot \nabla_x T\|^2_{H^s} \\ &\quad + \; \int_0^t \|w^* \, \partial_z v\|^2_{H^s} \; + \; \int_0^t \|w^* \cdot \partial_z \, T\|^2_{H^s} \; + \; \int_0^t \|\overline{w} \, \partial_z \, v\|^2_{H^s} \\ &\quad + \; \int_0^t \|\overline{w} \, \partial_z \, T\|^2_{H^s} \; + \; \varepsilon^{2n} \end{aligned}$$

As $s > 3/2$, H^s is a Banach algebra, so that:

$$\begin{aligned} \int_0^t \|v \cdot \nabla_x v\|^2_{H^s} \; &\leq \; \int_0^t \|v\|^2_{H^s} \, \|\nabla_x v\|^2_{H^s} \\ &\leq \; \varepsilon^{-2} \left(\varepsilon \sup_{0 \leq t' \leq t} \|v(t')\|^2_{H^s} \right) \left(\varepsilon \int_0^t \|\nabla_x v\|^2_{H^s} \right) \\ &\leq \; \varepsilon^{-2} \alpha^\varepsilon(t)^2. \end{aligned}$$

Similarly,

$$\int_0^t \|v\cdot\nabla_x T\|_{H^s}^2 \;\le\; \varepsilon^{-1}\left(\varepsilon \sup_{0\le t'\le t}\|v(t')\|_{H^s}^2\right)\int_0^t \|\nabla_x T\|_{H^s}^2$$
$$\le\; \varepsilon^{-1}\alpha^\varepsilon(t)^2.$$

We remind that $w^* = -(\int_0^z \mathrm{div}_{\,x} v)^*$, hence

$$\|w^*\,\partial_z v\|_{H^s} \;\lesssim\; \|\nabla v\otimes\nabla v\|_{H^s}^2.$$

As in section 4, we use the Gagliardo-Nirenberg inequality to deduce

$$\|\nabla v\otimes\nabla v\|_{H^s} \;\lesssim\; \|\nabla v\|_{L^\infty}\,\|\nabla v\|_{H^s}$$
$$\lesssim\; \|v\|_{H^s}\,\|\nabla v\|_{H^s}$$

We get as before

$$\int_0^t \|w^*\,\partial_z v\|_{H^s}^2 \;\lesssim\; \varepsilon^{-2}\alpha^\varepsilon(t)^2.$$

In the same way, we obtain

$$\|w^*\,\partial_z T\|_{H^s} \;\lesssim\; (\|\nabla v\|_{L^\infty} + \|\nabla T\|_{L^\infty})\,(\|\nabla v\|_{H^s} + \|\nabla T\|_{H^s})$$
$$\lesssim\; (\|v\|_{H^s} + \|T\|_{H^s})\,(\|\nabla v\|_{H^s} + \|\nabla T\|_{H^s})$$

which implies

$$\int_0^t \|w^*\,\partial_z T\|_{H^s}^2 \;\lesssim\; \varepsilon^{-2}\alpha^\varepsilon(t)^2.$$

Once again, we can express $\overline{w}$ in terms of v^* and T through:

$$\overline{w} = -\overline{u_{app}\cdot\nabla T} \;-\; \overline{u\cdot\nabla T_{app}} \;-\; \overline{u\cdot\nabla T} \;+\; \overline{R_{app,T}}.$$

Still using the Gagliardo-Nirenberg inequality, we end up with

$$\|\overline{w}\,\partial_z v\|_{H^s} \;\lesssim\; \Big(\|v\|_{W^{1,\infty}} + \|T\|_{W^{1,\infty}}$$
$$+\; \|v\|_{W^{1,\infty}}^2 + \|T\|_{W^{1,\infty}}^2 + \varepsilon^n\Big)\,(\|\nabla v\|_{H^s} + \|\nabla T\|_{H^s})$$
$$\lesssim\; \big(\|v\|_{H^s} + \|T\|_{H^s} + \|v\|_{H^s}^2 + \|T\|_{H^s}^2 + \varepsilon^n\big)\,(\|\nabla v\|_{H^s} + \|\nabla T\|_{H^s})$$

and

$$\int_0^t \|\overline{w}\,\partial_z v\|_{H^s}^2 \;\lesssim\; \varepsilon^{-2}\alpha^\varepsilon(t)^2 + \varepsilon^{-3}\alpha^\varepsilon(t)^3 + \varepsilon^{2n-1}\alpha^\varepsilon(t)$$
$$\lesssim\; \varepsilon^{-2}\alpha^\varepsilon(t)^2 + \varepsilon^{-3}\alpha^\varepsilon(t)^3 + \varepsilon^{4n-2}$$

for ε small enough. In the same way:

$$\int_0^t \|\overline{w}\partial_z T\|_{H^s}^2 \lesssim \varepsilon^{-2}\alpha^\varepsilon(t)^2 + \varepsilon^{-3}\alpha^\varepsilon(t)^3 + \varepsilon^{4n-2}.$$

Back to the energy estimates, it finally yields

$$\alpha^\varepsilon(t) \lesssim \varepsilon^{2n} + \varepsilon^{-2}\alpha^\varepsilon(t)^2 + \varepsilon^{-3}\alpha^\varepsilon(t)^3$$

with $\alpha^\varepsilon(0) = 0$. It implies straightforwardly that, for ε small enough, for all $t \in [0, \min(\tau^\varepsilon, \tau_*)]$,

$$\alpha^\varepsilon(t) \leq \varepsilon^n.$$

In particular, we have $\tau^\varepsilon \geq \tau_*$, which provides the first part of the theorem. Then, we deduce

$$\varepsilon^{-1} \sup_{0\leq t\leq \tau_*} \alpha^\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} 0,$$

which ends the proof.

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