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Three-Dimensional Instability of Planar Flows

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Abstract

We study the stability of two-dimensional solutions of the three-dimensional Navier–Stokes equations, in the limit of small viscosity. We are interested in steady flows with locally closed streamlines. We consider the so-called elliptic and centrifugal instabilities, which correspond to the continuous spectrum of the underlying linearized Euler operator. Through the justification of highly oscillating Wentzel–Kramers–Brillouin expansions, we prove the nonlinear instability of such flows. The main difficulty is the control of nonoscillating and nonlocal perturbations issued from quadratic interactions.

1. Introduction

We consider the incompressible Navier–Stokes equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \frac{1}{Re} \Delta u + F, \quad \nabla \cdot u = 0 \tag{1}$$

in the regime of large Reynolds number Re, in the domain $\mathcal{D} = \Sigma \times \mathbb{T}$, where Σ is a two-dimensional domain without boundary ($\mathbb{R}^2$ or $\mathbb{T}^2$ for example). The source term F is an exterior forcing. Since we are interested in the dynamics at large Reynolds numbers, we set for convenience

$$\frac{1}{Re} = \nu \varepsilon^2,$$

where $\nu > 0$ is fixed and $\varepsilon > 0$ will be a small parameter. We shall study the instability of stationary solutions $\mathbf{u^s}$ of (1) which are two-dimensional, that is $\mathbf{u}^s(x) = (u^s(x_1, x_2), 0)$ when they are submitted to three-dimensional perturbations. This is a classical problem, with a long history in fluid mechanics. A particularly interesting class of stationary solutions is made of the two-dimensional stationary solutions of the Euler equation. In general they are not solutions of the homogeneous Navier–Stokes equation, so in this case, the exterior forcing F is used to prevent the slow motion induced by the small viscosity.

Let us consider a perturbation v of a stationary solution $\mathbf{u}^s$ that will evolve according to

$$\partial_t v + \mathbf{u}^s \cdot \nabla v + v \cdot \nabla \mathbf{u}^s + \nabla p + v \cdot \nabla v = \nu \varepsilon^2 \Delta v, \quad \nabla \cdot v = 0. \tag{2}$$

If the perturbation v is small, we can in a first step neglect the nonlinear term and only study the linear equation

$$\partial_t v + \mathbf{u}^s \cdot \nabla v + v \cdot \nabla \mathbf{u}^s + \nabla p = \nu \varepsilon^2 \Delta v, \quad \nabla \cdot v = 0. \tag{3}$$

The linear instability of $\mathbf{u}^s$ is linked to the existence of unstable spectrum for the linear operator

$$L^\varepsilon v = \mathbb{P}\big(- \mathbf{u}^s \cdot \nabla v - v \cdot \nabla \mathbf{u}^s + \nu \varepsilon^2 \Delta v \big),$$

where $\mathbb{P}$ is the Leray projection on divergence free vector fields. For the linearized Euler operator (that is, when $\nu = 0$), a classical way to find linear instabilities which was introduced by LIFSCHITZ—HAMEIRI [16] and FRIEDLANDER—VISHIK [8] is to study high-frequency oscillations: we look for solutions of (3) under the form

$$v = a(t,x) \mathrm{e}^{\mathrm{i} \frac{\varphi(t,x)}{\varepsilon}}. \tag{4}$$

The phase φ and the amplitude a at leading order solve the equations

$$\partial_t \varphi + \mathbf{u}^s \cdot \nabla \varphi = 0, \tag{5}$$

$$\partial_t a + \mathbf{u}^s \cdot \nabla a + \left(I - 2 \frac{\nabla \varphi \otimes \nabla \varphi}{|\nabla \varphi|^2} \right) a \cdot \nabla \mathbf{u}^s = 0 \tag{6}$$

together with the constraint $a \cdot \nabla \varphi = 0$. We notice that this last constraint remains verified if it is verified at $t = 0$. By using Lagrangian coordinates, we get the system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathbf{u}^s(x), \tag{7}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\nabla \mathbf{u}^s \cdot \xi, \tag{8}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(2 \frac{\xi \otimes \xi}{|\xi|^2} - I \right) D\mathbf{u}^s \cdot a \tag{9}$$

where $\xi(t) = \nabla \varphi(t, x(t, x_0))$. To avoid confusion, we recall that Du stands for the matrix $(\partial_j u_i)_{i,j}$ whereas $\nabla u = {}^t Du$. It was rigorously proved [26] that the existence of a positive Lyapunov exponent for the bicharacteristic-amplitude system (7)–(9) implies linear instability for (3). More recently, it was shown that the width of the essential spectrum of L^0 is determined by the Lyapunov exponent of the bicharacteristic-amplitude system [22, 25]. In particular, since this Lyapunov exponent is positive for flows with hyperbolic stagnation points (see [8]), this implies their linear instability. Nevertheless, there are other classical examples in fluid mechanics where the bicharacteristic-amplitude system can be analyzed. In the case where u^s is a stationary solution of the two-dimensional Euler equation, we can assume that the vorticity $\omega^s = \partial_1 u_2^s - \partial_2 u_1^s$ and the streamfunction ψ^s such that $u^s = \nabla^\perp \psi^s$ are linked through the relation $\omega^s = F(\psi^s)$. If u^s has closed streamlines, both the geometry of the streamlines and the distribution of the vorticity through the properties of F may contribute to some instability in the bicharacteristic-amplitude system. These are the well-known elliptic and centrifugal instabilities. From a physical point of view, the elliptic instability has been analyzed as an inertial wave resonance forced by the local strain induced by the ellipticity of the flow; for a review, we refer to [15]. The centrifugal instability discovered by Rayleigh [21] results from a stratification in angular momentum and is the analogue to the buoyancy-induced Rayleigh–Taylor instability for variable density incompressible fluids. The aim of this paper is to prove that these linear instabilities imply a nonlinear instability for Equation (1) when ε is sufficiently small.

Let us first describe the elliptic instability. We assume that there is a stagnation point x_0 such that $A = D\mathbf{u}^s(x_0)$ has purely imaginary eigenvalues. In this case, in the vicinity of the stagnation point, the flow $\mathbf{u}^s$ is well approximated by the linear flow Ax. By using this approximation we get a simpler bicharacteristic system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax, \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = -A^*\xi, \\ \dfrac{\mathrm{d}a}{\mathrm{d}t} = \left(2\dfrac{\xi \otimes \xi}{|\xi|^2} - I\right) A\, a. \end{cases} \tag{10}$$

Note that in this approximation, the vorticity is constant in the vicinity of the stagnation point so that the only ingredient which may contribute to the instability is the geometry of the streamlines. Due to this special setting, the system (10) is easier to analyze. Indeed, we observe that the first equation is decoupled from the others; moreover the gradient of the phase, ξ, is just given by $\xi(t) = \mathrm{e}^{-A^*t}\xi_0$ and is periodic in time. Thanks to these observations, the equation for the amplitude [the third equation of (10)] is a linear ordinary differential equation with periodic coefficients and hence can be analyzed with Floquet theory. We notice that the average over a period of the trace of the matrix $(2\xi \otimes \xi/|\xi|^2 - I)A$ is zero so that the sum of the Floquet exponents vanishes. Since the equation for a_3 is just $\frac{\mathrm{d}a_3}{\mathrm{d}t} = 0$, if a_h vanishes, we finally get that the Floquet exponents are given by $\zeta_0(\xi_0)$, $-\zeta_0(\xi_0)$ and 0. Our assumption of linear elliptic instability will be

(H1) There exists ξ_0 such that the real part $\sigma_0(\xi_0)$ of $\zeta_0(\xi_0)$ is positive.

This assumption has been checked numerically in fluid mechanics papers [2, 20] and was later checked analytically by Waleffe [27]: (H1) is actually verified for all elliptical noncircular flows. The details will be given in Section 4.

The other classical example that we want to analyze is the centrifugal instability. We still assume that u^s has locally closed streamlines, but we now study the dynamics in the vicinity of a closed streamline which is not necessarily close to the stagnation point. We introduce a local curved coordinate system

$$(\rho, \theta) \to y(\rho, \theta) = (x_1(\rho, \theta), x_2(\rho, \theta))$$

for $\rho \in (\rho_0 - \delta, \rho_0 + \delta)$ such that the streamfunction ψ^s defined as usual by $u^s = \nabla^\perp \psi^s$ is constant on the curves $\theta \mapsto y(\rho, \theta)$, that is

$$\psi^s(y(\rho, \theta)) = \rho.$$

Moreover, since we assume that the streamlines are locally closed, we have that $\theta \to y(\rho, \theta)$ is $T(\rho)$ periodic. Precisely, we choose a parameterization $\theta \mapsto y_0(\theta)$ of the streamline $\rho = \rho_0$, and $y(\rho, \theta)$ satisfying the ordinary differential equation (ODE)

$$\partial_\rho y(\rho, \theta) = \frac{\nabla \psi_s(y(\rho, \theta))}{|\nabla \psi_s(y(\rho, \theta))|^2}, \quad y(\rho_0, \theta) = y_0(\theta).$$

A local orthonormal basis (e_ρ, e_θ) is then given by

$$e_\theta = \frac{1}{h(\rho, \theta)} \partial_\theta y, \quad e_\rho = |u^s| \partial_\rho y.$$

In this coordinate system, u^s has the shape

$$u^s(y) = U^s(\rho, \theta) e_\theta \quad \forall \rho \in (\rho_0 - \delta, \rho_0 + \delta). \tag{11}$$

Thanks to this structural assumption on our basic flow, we can reduce the study of the Lyapunov exponent of the amplitude equation (9). Indeed, we first notice that Equation (7) reduces to

$$\theta'(t) = \frac{U^s}{h}(\theta(t), \rho), \quad \rho'(t) = 0.$$

Since U^s is periodic, θ will be $T(\rho)$ periodic. Next, we restrict our study to the simple phase

$$\xi(\rho, \theta(t)) = e_z \tag{12}$$

With this special choice of the phase, we can rewrite the equation for the amplitude as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = G(t, \theta_0, \rho) a. \tag{13}$$

Again, the Floquet exponents for this system are $\zeta_0(\rho)$, $-\zeta_0(\rho)$, 0 (we can easily check that they do not depend on θ_0 because of the periodicity of θ). A streamline ρ_0 will be said to be centrifugally unstable if $\sigma_0(\rho_0) = Re\,(\zeta_0)$ is positive. Our assumption for linear centrifugal instability reads

(H1′) There exists ρ_0 such that $\sigma_0(\rho_0) > 0$.

Again this assumption can be checked numerically, see [23], for example. Moreover, there are some criteria that allow to predict the presence of centrifugal instability [1, 3]. Various examples will be given in Section 4. In some specific situations, explicit computations are possible. In the case of circular vortices $u^s = U^s(r)e_\theta$ where (r, θ) are the standard polar coordinates in the plane, the matrix G does not depend on time and hence we just have to compute its eigenvalues. This yields the famous Rayleigh criterion for centrifugal instability (see Section 4): the flow is unstable if the sign of the vorticity changes. This inherently three-dimensional instability differs from the two-dimensional azimuthal shear instability. A necessary condition for this purely two-dimensional shear instability, also due to Rayleigh, is the extension to circular geometry of the inflection point theorem and requires a change in the sign of $\mathrm{d}W(r)/\mathrm{d}r$ where W is the vorticity. Consequently, we see on this simple example that there are flow which are stable in two dimensions but unstable in three dimensions.

The aim of this paper is to prove that assumption (H1) or assumption (H1′) implies a localized nonlinear instability for Equation (1). Of course, the main tool will be the justification of weakly nonlinear Wentzel–Kramers–Brillouin (WKB) expansions like (4) for the nonlinear equation (1) on sufficiently long times. Since we are looking for real-valued solution of the nonlinear equation (1) we shall look for solutions under the form

$$u = \mathbf{u}^s + \varepsilon^{N+1}\left(a\mathrm{e}^{\frac{\mathrm{i}\varphi}{\varepsilon}} + \overline{a}\mathrm{e}^{\frac{-\mathrm{i}\varphi}{\varepsilon}}\right) + \varepsilon^{2N+1}v^p\left(t, x_1, x_2, \frac{\varphi}{\varepsilon}\right),$$

where φ, a are given by the bicharacteristic amplitude system and v^p is a perturbation. More precisely, φ is given by (7), (8) and a is, in the presence of viscosity, given by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - I\right) D\mathbf{u}^s \cdot a - \nu|\xi|^2 a.$$

Because of (H1) or (H1′) we can find an unstable solution a such that

$$a \sim \mathrm{e}^{\zeta t}, \quad \sigma = \mathrm{Re}\,\zeta > 0$$

for ν sufficiently small since the Floquet exponents depend continuously on ν. Indeed, here we have explicitly

$$\sigma = \sigma_0 - \alpha\nu, \quad \alpha = \frac{1}{T}\int_0^T |\xi|^2,$$

where σ_0 is given by (H1) or (H1′). When we plug this ansatz into the nonlinear equation (1), the interaction of the two oscillating modes generates a term under the form $a \cdot \nabla\overline{a}$ which is not oscillating and in particular does not depend on x_3. Consequently the average of the horizontal components of v^p will solve at leading order a two-dimensional linearized Navier–Stokes equation

$$\partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + \nabla p - \nu\varepsilon^2 \Delta v = f(t, x_1, x_2), \quad \nabla \cdot v = 0 \tag{14}$$

with a source term that behaves like $\mathrm{e}^{2\sigma t}$. To prove an instability result, we shall need on a sufficiently long time scale that $\varepsilon^{2N+1}v^p$ remains smaller than the term

$\varepsilon^{N+1}(a\mathrm{e}^{\frac{\mathrm{i}\varphi}{\varepsilon}} + \overline{a}\mathrm{e}^{\frac{-\mathrm{i}\varphi}{\varepsilon}})$ which describes the linear instability. This will be true if we can find a solution of (14) which remains bounded by $\mathrm{e}^{2\sigma t}$. In other words, the three-dimensional instability which is detected by the bicharacteristic amplitude system should be stronger than the possible two-dimensional instabilities in (14). To formalize this assumption, let us introduce a space X made of smooth functions of two variables, which contains the unstable solution of the amplitude equation (6) given by (H1) or (H1′) and is stable by derivation and multiplication. We shall denote by $||\cdot||$ the L^2 norm. Then our second assumption reads

(H2) There exists $C > 0$ such that for every $\varepsilon \in (0, 1)$ and for every $f(t, x_1, x_2) \in X$ such that

$$\|f\| + \|\mathrm{curl} f\| \leqq C_f \mathrm{e}^{\gamma t} \tag{15}$$

with $\gamma \geqq 2\sigma$ then the two-dimensional solution v of

$$\partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + \nabla p - \nu\varepsilon^2 \Delta v = f, \quad \nabla \cdot v = 0, \tag{16}$$

which vanishes at $t = 0$ is in X and satisfies the estimate

$$\|v\| + \|\mathrm{curl} v\| \leqq C C_f \mathrm{e}^{\gamma t}. \tag{17}$$

This assumption means that the instability that we have detected dominates the two-dimensional instability in a certain class X. This assumption is rather difficult to check analytically, a few examples of flows for which it is possible are given in Section 4. The meaning of the space X is that we can use the symmetries of the problem to check (H2). For example, in the case of the centrifugal instability of the circular vortices $u^s(r) = U(r)e_\theta$ in polar coordinates, we can choose X as the space of smooth functions in the vicinity of r_0 that are independent of θ. Note that in this space we cannot see the two-dimensional instabilities: in this case, the solutions of (16) also depend only on r, so that (16) reduces to the heat equation

$$\partial_t v - \nu\varepsilon^2 \Delta v = f$$

and hence the assumption (H2) is clearly verified. More generally, the competition between the centrifugal instability and the shear instability has been studied numerically in details in [4] and it was seen that the centrifugal instability dominates the two-dimensional shear waves, so that our assumption (H2) seems to be generically verified.

We are now able to state our main results. We begin with the nonlinear elliptic instability:

Theorem 1. (Nonlinear elliptic instability) *Consider* $\mathbf{u}^s(x) = (u_s(x_1, x_2), 0)$ *a stationary solution of* (1) *and assume that there exists a stagnation point* x_0 *such that* $A = D\mathbf{u}^s(x_0)$ *has only purely imaginary eigenvalues. Then if* $\Sigma = \mathbb{R}^2$ *or* $\mathbb{T}^2$ *and under the assumptions* (H1) *and* (H2), $\mathbf{u}^s$ *is nonlinearly unstable in the following sense:*

there exists ν_0 *such that, for every* $\nu \in (0, \nu_0]$ *and for every* $N \in \mathbb{N}$, $s \in \mathbb{N}$ *and* $\beta \in (1/3, 1)$, *there exists* ε_0 *and* $\eta > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$, *there exists an initial data* $v^{0,\varepsilon}$ *and a time* T^ε *such that*

$$\|v^{0,\varepsilon}\|_{H^s} \leqq \varepsilon^N$$

and the solution of Equation (2) *verifies*

$$\|v^\varepsilon(T^\varepsilon)\|_{L^2(D(x_0,\varepsilon^\beta)\times\mathbb{T})} \geqq \eta\varepsilon^{\beta+1}.$$

Moreover, we also have

$$\sup_{t\in[0,T^\varepsilon]} \|v^\varepsilon(t)\|_{L^\infty(D(x_0,\varepsilon^\beta)\times\mathbb{T})} \geqq \eta\varepsilon.$$

This theorem is a nonlinear instability result: it states that a small perturbation u^ε can be amplified from the amplitude ε^N, for arbitrary N, until it reaches the amplitude $\varepsilon^{\beta+1}$ in the L^2 norm and ε in the L^∞ norm locally in the small disk $D(x_0, \varepsilon^\beta)$ around the stagnation point. Note that in the same disk $D(x_0, \varepsilon^\beta)$, we have that

$$\|\mathbf{u}^s\|_{L^2(D(x_0,\varepsilon^\beta)\times\mathbb{T})} \leqq C\varepsilon^{2\beta}, \quad \|\mathbf{u}^s\|_{L^\infty(D(x_0,\varepsilon^\beta)\times\mathbb{T})} \leqq C\varepsilon^{\beta}.$$

Since β can be chosen arbitrarily close to 1, this yields that locally the perturbation u^ε can almost reach the amplitude of the reference flow $\mathbf{u}^s$. In this way we recover a classical instability result. There was a lot of recent activity towards the justification of the fact that linear instability implies nonlinear instability for the Euler or Navier–Stokes equation, see [7, 11, 14, 17]. In particular it was established in [17] that for the two-dimensional Euler equation the existence of an unstable eigenvalue implies nonlinear instability. Moreover, the importance of the norm in which the instability is stated has been emphasized in [17]: there exists flows that are linearly unstable in the L^2 norm of the velocity but stable in the L^2 norm of the vorticity. Here our result has a simple and clear physical interpretation: there is a violent amplification of the amplitude of the perturbation of the velocity in the L^2 and Sup norm in the vicinity of the stagnation point. Finally, note that our result of nonlinear instability does not rely on the existence of an unstable eigenvalue for the linearized operator, but on the use of the high-frequency expansions (46) to detect instabilities. In this way our analysis is closer to the one in [6]; see also [9, 10] for applications in fluid dynamo theory. Indeed, we shall look as in [6] for a high-order WKB expansion. The main technical difficulty of the paper is that we shall have to control the growth of high-order derivatives of the solutions of (16), in order to construct a high-order WKB expansion. For the linearized Euler equation, generally, the derivatives have a faster growth. Indeed, if u^s has a hyperbolic stagnation point, the derivative $\partial^\alpha \mathrm{curl} v$ has a growth $\mathrm{e}^{|\alpha|\mu t}$. This is why we keep the viscosity and study the solutions of (16) and not the solutions of the linearized Euler equation. We shall establish that for the solutions of (16) the estimate (17) is still true for penalized derivatives $\varepsilon^{|\alpha|}\partial^\alpha$, whereas locally in the vicinity $D(x_0, \varepsilon^\beta)$ of the stagnation point, it is possible to control $\varepsilon^{\beta|\alpha|}\partial^\alpha$.

In a similar way, we can prove nonlinear centrifugal instability:

Theorem 2 (Nonlinear centrifugal instability)**.** *Consider* $\mathbf{u}^s(x) = (u_s(x_1, x_2), 0)$ *a two-dimensional stationary solution of* (1) *and assume that there exists closed streamlines in the vicinity of* ρ_0*. Then if* $\Sigma = \mathbb{R}^2$ *or* $\mathbb{T}^2$ *and under the assumption* (H1$'$) *and* (H2)*,* $\mathbf{u}^s$ *is nonlinearly unstable in the following sense: there exists* ν_0 *such that for every* $\nu \in (0, \nu_0]$ *and for every* $N \in \mathbb{N}$*,* $s \in \mathbb{N}$*, there exists* ε_0 *and* $\eta > 0$ *such for every* $\varepsilon \in (0, \varepsilon_0)$ *there exists an initial data* $v^{0,\varepsilon}$ *and a time* T^ε *such that*

$$\|v^{0,\varepsilon}\|_{H^s} \leqq \varepsilon^N$$

and the solution of Equation (2) *verifies*

$$\|v^\varepsilon(T^\varepsilon) \cdot e_\rho\|_{L^2(\rho_0-\sqrt{\varepsilon},\rho_0+\sqrt{\varepsilon})} \geqq \eta\varepsilon^{\frac{5}{4}}.$$

Moreover, we also have

$$\sup_{t\in[0,T^\varepsilon]} \|v^\varepsilon(t) \cdot e_\rho\|_{L^\infty(\rho_0-\sqrt{\varepsilon},\rho_0+\sqrt{\varepsilon})} \geqq \eta\varepsilon.$$

This theorem states that the perturbation v^ε normal to the streamlines of u^s can be arbitrarily amplified.

The main difficulty to apply these theorems is to check the assumptions (H1) or (H1$'$) and (H2). This will be done in the second part of the paper on various classical flows of fluid mechanics.

The paper is organized as follows: the first part is devoted to the detailed proof of Theorem 1. The second one is devoted to the proof of Theorem 2. Finally the last part is devoted to the application of the previous theorems to specific examples like Taylor–Green vortices, or parallel and circular shear flows with Coriolis forcing. To lighten notations, we will drop the bold symbol $\mathbf{u}^s$: in the following, u^s will denote either the two-dimensional flow $u^s(y) \in \mathbb{R}^2$ or its three-dimensional counterpart $u^s(x) \in \mathbb{R}^3$ with $\partial_3 u^s = 0$, $u^s_3 = 0$. The same convention will hold for $A = Du^s$. Distinction will be clear from the context.

2. Elliptic instability

2.1. WKB expansion

The aim of this section is to prove Theorem 1. At first, we set $u = u^s + \varepsilon v$ so that v solves

$$\partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + \nabla p = \nu\varepsilon^2 \Delta v - \varepsilon v \cdot \nabla v, \quad \nabla \cdot v = 0 \tag{18}$$

Without loss of generality, we assume that the stagnation point $x_0 = 0$, so that we can write

$$u^s = Ay + v^s(y), \quad v^s = \mathcal{O}(|y|^2)$$

in a vicinity of the stagnation point. We set $U = (v, p)$, and we rewrite (18) as

$$L^\varepsilon U + \varepsilon v \cdot \nabla v = 0, \quad \nabla \cdot v = 0. \tag{19}$$

where

$$L^\varepsilon U = \partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + \nabla p - \nu\varepsilon^2 \Delta v.$$

The first step is to look for an expansion under the form

$$U^{\text{app}} = \varepsilon^N \sum_{k=0}^{M} \varepsilon^{Nk} U^k, \quad U^k = (V^k, P^k). \tag{20}$$

Plugging this ansatz into (19), we find

$$\sum_{k=0}^{M} \varepsilon^{Nk} L^\varepsilon U^k + \varepsilon^{N+1} \sum_{0 \leqq k,l \leqq M} \varepsilon^{N(k+l)} V^k \cdot \nabla V^l = 0,$$

$$\sum_{k=0}^{M} \varepsilon^{Nk} \operatorname{div} V^k = 0.$$

We solve approximately this system by setting

$$\begin{cases} L^\varepsilon U^0 = \varepsilon^N R^0, \\ \operatorname{div} V^0 = \varepsilon^N d^0 \end{cases} \tag{21}$$

and for $k \geqq 1$

$$\begin{cases} L^\varepsilon U^k = -R^{k-1} + \varepsilon^N R^k + \varepsilon \displaystyle\sum_{j+l=k-1} V^j \cdot \nabla V^l, \\ \operatorname{div} V^k = -d^{k-1} + \varepsilon^N d^k. \end{cases} \tag{22}$$

The terms R^k, $k \geqq 0$ will be uniformly bounded with respect to ε on a sufficiently large interval of time $[0, T_\varepsilon]$ for ε sufficiently small. If we had a complete understanding of the linear operator L^ε we could solve the system with $R^k = 0$. Since this is not the case here, we will use a WKB expansion of U^k to solve (21) and (22).

2.2. Resolution of (21)*; choice of an unstable solution*

We set

$$\beta = 1 - \frac{1}{\mu}$$

with $\mu \geqq 3$. We look for a WKB expansion U^0 under the form

$$U^0(t, x) = \mathcal{U}^0\left(t, \frac{y}{\varepsilon^\beta}, \frac{\varphi(t, x/\varepsilon^\beta)}{\varepsilon^{\frac{1}{\mu}}}\right),$$

where $\mathcal{U}(t, Y, \lambda) = (\mathcal{V}(t, Y, \lambda), \mathcal{P}(t, Y, \lambda))$ is compactly supported in $D(0, 1) \subset \mathbb{R}^2$ in Y and periodic in λ. Note that U^0 depends on the x_3 variable only through the phase. Moreover, we also require that

$$\int_\lambda \mathcal{U}^0(t, Y, \lambda)\, \mathrm{d}\lambda = 0. \tag{23}$$

Choice of the phase We want φ to be a solution of

$$\partial_t \varphi + AY \cdot \nabla_Y \varphi = 0. \tag{24}$$

The choice

$$\varphi(t, X) = \mathrm{e}^{-tA} Y \cdot \xi_h^0 + Z\, \xi_3^0 \tag{25}$$

where $\xi^0 = (\xi_h^0, \xi_3^0)$ is given by (H1) will be sufficient for our purpose. Note that $\nabla\varphi$ depends on time only and also since we are studying an elliptic stagnation point that $t \mapsto \|\mathrm{e}^{tA}\|$ is uniformly bounded.

Thanks to this choice we get for every α the following estimate for the phase:

$$|\partial_Y^\alpha \nabla_X \varphi(t, X)| \leqq C_\alpha \quad \forall t \geqq 0\ \forall Y \in D(0, 1). \tag{26}$$

Choice of the profiles We look for an expansion of $\mathcal{U}$ under the form

$$\mathcal{U}^0 = \sum_{l=0}^{\mu(N+1)} \varepsilon^{\frac{l}{\mu}} \mathcal{U}^{0,l}\left(t, \frac{y}{\varepsilon^\beta}, \frac{\varphi(t, x/\varepsilon^\beta)}{\varepsilon^{\frac{1}{\mu}}}\right), \tag{27}$$

where

$$\mathcal{U}^{0,l} = \left(\mathcal{V}^{0,l}, \varepsilon \mathcal{P}^{0,l}\right)$$

is periodic in λ, verifies (23), and is compactly supported in $D(0, 1)$ in Y. For $Y \in D(0, 1)$, we can use the Taylor expansion of u^s given by

$$u^s(\varepsilon^\beta Y) = \varepsilon^\beta AY + \mathcal{O}(\varepsilon^{2\beta}).$$

Since $\beta = (\mu - 1)/\mu$, we can plug the expansion (27) into (21) and identify the powers of $\varepsilon^{\frac{1}{\mu}}$. The first term, which is the ε^{-1} term in the divergence-free condition, gives that $\partial_\lambda \mathcal{V}^{0,0} \cdot \nabla\varphi = 0$. Thanks to the condition (23) this yields

$$\mathcal{V}^{0,0} \cdot \nabla\varphi = 0. \tag{28}$$

Then, thanks to the choice of the phase, the ε^0 term in the first line of (21) gives

$$\partial_t \mathcal{V}^{0,0} + AY \cdot \nabla_Y \mathcal{V}^{0,0} + A\mathcal{V}^{0,0} - \nu|\nabla\varphi|^2\, \partial^2_{\lambda\lambda} \mathcal{V}^{0,0} + \partial_\lambda \mathcal{P}^{0,0}\, \nabla\varphi = 0.$$

By taking the scalar product by $\nabla\varphi$ and by using (28), we actually get from the last equation

$$|\nabla\varphi|^2\, \partial_\lambda \mathcal{P}^{0,0} = 2A\mathcal{V}^{0,0} \cdot \nabla\varphi, \tag{29}$$

$$\partial_t \mathcal{V}^{0,0} + AY \cdot \nabla_Y \mathcal{V}^{0,0} = M(\nabla\varphi)\mathcal{V}^{0,0} + \nu|\nabla\varphi|^2 \partial^2_{\lambda\lambda} \mathcal{V}^{0,0}, \tag{30}$$

where

$$\mathcal{M}(\xi)a = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - I\right) A.$$

Note that the first line determines in a unique way the pressure with the condition (23). Moreover, the $\varepsilon^{-1+\frac{1}{\mu}}$ term in the divergence-free condition in (21) gives

$$\partial_\lambda \mathcal{V}^{0,1} \cdot \nabla\varphi = -\nabla_Y \cdot \mathcal{V}^{0,0}$$

which determines $\mathcal{V}^{0,1} \cdot \nabla\varphi$ with respect to $\mathcal{V}^{0,0}$ in a unique way again thanks to (23).

In the same way, for $k \geqq 1$, we get that $(\mathcal{V}^{0,k}, \mathcal{V}^{0,k+1} \cdot \nabla\varphi^0, \mathcal{P}^{0,k})$ solves the system

$$\partial_\lambda \mathcal{V}^{0,k+1} \cdot \nabla\varphi = D^{0,k}, \tag{31}$$

$$|\nabla\varphi|^2\, \partial_\lambda \mathcal{P}^{0,k} = 2A\mathcal{V}^{0,k} \cdot \nabla\varphi + L^{0,k} \cdot \nabla\varphi, \tag{32}$$

$$\partial_t \mathcal{V}^{0,k} + AY \cdot \nabla_Y \mathcal{V}^{0,k} = \mathcal{M}(\nabla\varphi)\mathcal{V}^{0,k} - \nu|\nabla\varphi|^2 \partial^2_{\lambda\lambda} \mathcal{V}^{0,k} + \mathcal{P}(\nabla\varphi)L^{0,k}, \tag{33}$$

where $L^{0,k}$ only depends on $(\mathcal{V}^{0,j}, \mathcal{V}^{0,j+1} \cdot \nabla\varphi, \mathcal{P}^{0,j})_{j \leqq k-1}$ and on $\nabla\varphi$. Also $D^{0,k}$ only depends on $(\mathcal{V}^{0,j})_{j\leqq k}$ and on $\nabla\varphi$.

Our aim is to use (H1) to find a growing solution of (30) and then to estimate uniformly the correction terms which solve (31)–(33). We set

$$T_\varepsilon = \frac{N|\log\varepsilon|}{\sigma},$$

where $\sigma = \sigma_0 - \alpha\nu$ with σ_0 given by (H1). All our estimates will be uniform on $[0, T_\varepsilon]$ for ε sufficiently small. In the following all the numbers denoted by C which change from lines to lines may depend on N, μ, and M but are independent of ε.

We gather the results in the following lemma:

Lemma 3. *Thanks to* (H1)*, there exists* $\mathcal{U}^0$ *which solves* (21) *smooth and such that we have*

$$e^{\sigma t} \leqq \|\mathcal{V}^0(t)\| \leqq C e^{\sigma t}, \quad t \in [0, T_\varepsilon]. \tag{34}$$

Moreover, there exists ε_0 *such that for every* ε, $0 < \varepsilon \leqq \varepsilon_0$*, and for every* α *we have the estimate:*

$$|\partial_Y^\alpha \mathcal{U}^0(t, Y, \lambda)| \leqq C_{\alpha,N} e^{\sigma t} \quad \forall Y \in D(0, 1), \tag{35}$$

$$\left|\partial_Y^\alpha \left(\mathcal{V}^0(t, Y, \lambda) \cdot \nabla\varphi(t, X)\right)\right| \leqq C_{\alpha,N}\varepsilon^{\frac{1}{\mu}} e^{\sigma t} \quad \forall Y \in D(0, 1), \tag{36}$$

and

$$|\partial_Y^\alpha R^0(t, Y, \lambda)| + |\partial_Y^\alpha d^0(t, Y, \lambda)| \leqq C_{\alpha,N}\, e^{\sigma t} \tag{37}$$

for $t \in [0, T_\varepsilon]$.

We have used the notation $\|\mathcal{V}^0\|$ for the L^2 norm of the profile $\mathcal{V}^0(t, Y, \lambda)$. In the original variables, we get from (34) that

$$\varepsilon^\beta e^{\sigma t} \leqq \|V^0(t)\| \leqq C\varepsilon^\beta e^{\sigma t}, \quad t \in [0, T_\varepsilon]. \tag{38}$$

Note the improved estimate for $\mathcal{V}^0 \cdot \nabla\varphi$ in (36) which will be important in the following.

Proof. The first step is to choose a growing solution of (30). We shall use (H1): we look for $\mathcal{V}^{0,0}$ under the form

$$\mathcal{V}^{0,0}(t, Y, \lambda) = a(t, Y)e^{i\lambda} + \text{c.c.}$$

where c.c. means complex conjugate with a such that $a(0, Y) \cdot \nabla\varphi = 0$ so that (28) is matched. Next, we observe that $\nabla_Y\varphi^0(t, e^{tA}Y)$ does not depend on Y so that we can set $\xi(t) = \nabla_Y\varphi^0(t, e^{tA}Y)$. By using Lagrangian coordinates, the equation for a becomes, thanks to (30):

$$\frac{d}{dt}a(t, e^{tA}Y) = \left(\mathcal{M}(\xi(t)) - \nu|\xi(t)|^2\right) a(t, e^{tA}Y). \tag{39}$$

To solve this ordinary differential equation, we first notice that Y is only a parameter, so that we can set

$$a(t, e^{tA}Y) = b(t)\chi(Y), \tag{40}$$

where $\chi \in [0, 1]$ is any smooth, compactly supported function in $D(0, 1)$ and $b(t)$ is a solution of (39). By (H1), we know that the ODE $b' = \mathcal{M}(\xi(t))b$ has a positive Floquet exponent. Consequently, by continuity there exists ν_0 such that for $\nu \in (0, \nu_0)$ there is still a positive Floquet exponent for the ODE $b' = (\mathcal{M}(\xi(t)) - \nu|\xi(t)|^2)b$. Consequently, we can choose a solution of (39) under the form $b(t) = e^{\zeta t}R(t)$ with $\sigma = \operatorname{Re}\zeta > 0$ and R a T-periodic vector. Note that σ and R are independent of Y_0. Going back to a defined by (40), we finally get

$$a(t, Y) = e^{\sigma t}R(t)\chi(e^{-tA}Y). \tag{41}$$

Thanks to this choice, we easily get after a suitable renormalization that

$$2e^{\sigma t} \leqq \|\mathcal{V}^{0,0}(t)\| \leqq 4e^{\sigma t}. \tag{42}$$

Once, we have made this choice, it is very easy to solve system (31)–(33). The first equation determines in a unique way $\mathcal{V}^{0,k+1} \cdot \nabla\varphi^0$ with respect to $\mathcal{V}^{0,k}$ with the constraint (23). In a similar way, the second one determines $\mathcal{P}^{0,k}$. Finally, the last equation allows to determine $\mathcal{V}^{0,k}$. To solve the last one, we set

$$\mathcal{V}^{0,k} = a^k(t, Y)e^{i\lambda} + \text{c.c.}$$

Again, (33) can be reduced to an ODE for a^k by using Lagrangian coordinates. We choose the solution with zero initial value. We easily get by standard ODE theory that

$$|\partial_Y^\alpha a^k(t, Y)| \leqq C_\alpha t^k e^{\sigma t} \tag{43}$$

for every α.

Finally, since $\mathcal{U}^0$ is defined by (27), we easily get

$$|\partial_Y^\alpha \mathcal{U}^0| \leqq C_\alpha e^{\sigma t} \left(1 + \sum_{l \geqq 1} (\varepsilon^{\frac{1}{\mu}} T_\varepsilon)^l\right) \leqq C_\alpha e^{\sigma t}$$

for ε sufficiently small. This proves (35). The estimate (37) follows easily. Moreover, we note that

$$\mathcal{V}^0 \cdot \nabla \varphi = \sum_{l=1}^{\mu(N+1)} \varepsilon^{\frac{l}{\mu}} d^{0,l}$$

thanks to (28). Since $d^{0,l}$ depends only on $(\mathcal{V}^k)_{k \leqq l-1}$, (43) gives

$$\left|\partial_Y^\alpha (\mathcal{V}^0 \cdot \nabla \varphi)\right| \leqq C_\alpha e^{\sigma t} \sum_{l=1}^{\mu(N+1)} \varepsilon^{\frac{l}{\mu}} T_\varepsilon^{l-1} \leqq C_\alpha e^{\sigma t} \varepsilon^{\frac{1}{\mu}}$$

for $t \in [0, T_\varepsilon]$.

Finally, we also have

$$\|\mathcal{V}^0(t)\| \geqq \varepsilon^{1-\frac{1}{\mu}} e^{\sigma t} \left(2 - C \sum_{l \geqq 1} \varepsilon^{\frac{l}{\mu}} T_\varepsilon^l\right) \geqq \varepsilon^{1-\frac{1}{\mu}} e^{\sigma t}$$

for ε sufficiently small. □

2.3. Resolution of (22)

Once $\mathcal{U}^0$ is chosen like in the previous section, we now turn to the resolution of (22). we point out that, in the following construction, R^{k-1}, d^{k-1} will be chosen under the form $\mathcal{R}^{k-1}(t, Y, \lambda)$, $\mathcal{D}^{k-1}(t, Y, \lambda)$ such that (23) is verified. From the previous section, this is already true for $\mathcal{R}^0$ and $\mathcal{D}^0$. We look for U^k under the form

$$U^k(t, x) = \mathcal{U}^k\left(t, \frac{y}{\varepsilon^\beta}, \frac{\varphi}{\varepsilon^{\frac{1}{\mu}}}\right) + u^k(t, y) \tag{44}$$

where $\mathcal{U}^k(t, Y, \lambda) = (\mathcal{V}^k(t, Y, \lambda), \mathcal{P}^k(t, Y, \lambda))$ verifies (23). We also set $u^k = (\mathbf{v}^k, p^k)$. Note that $\mathbf{v}^k$ depends only on y but that $\mathbf{v}^k \in \mathbb{R}^3$. We use the notation $\mathbf{v}^k = (v^k, v_3^k)$, where the horizontal part v^k is in $\mathbb{R}^2$. From the choice made in the previous section, we have $u^0 = 0$. Let us define the nonlinear term in the right-hand side of (22) as

$$Q^k = \varepsilon \sum_{j+l=k-1} V^j \cdot \nabla V^l.$$

Thanks to the decomposition (44), we can rewrite it as

$$Q^k = Q^{k,1} + Q^{k,2} + Q^{k,3}$$

where

$$
\begin{aligned}
Q^{k,1} &= \varepsilon \sum_{j+l=k-1} \left(\mathcal{V}^j \cdot \nabla \mathbf{v}^l + \mathbf{v}^j \cdot \nabla \mathcal{V}^l \right) \\
&= \sum_{j+l=k-1} \left(\varepsilon \mathcal{V}^j \cdot \nabla \mathbf{v}^l + \mathbf{v}^j \cdot \nabla_X \varphi \, \partial_\lambda \mathcal{V}^l + \varepsilon^{\frac{1}{\mu}} \mathbf{v}^j \cdot \nabla_Y \mathcal{V}^l \right), \\
Q^{k,2} &= \varepsilon \sum_{j+l=k-1} \mathbf{v}^j \cdot \nabla \mathbf{v}^l, \\
Q^{k,3} &= \varepsilon \sum_{j+l=k-1} \mathcal{V}^j \cdot \nabla \mathcal{V}^l \\
&= \sum_{j+l=k-1} \mathcal{V}^j \cdot \nabla_X \varphi \, \partial_\lambda \mathcal{V}^l + \varepsilon^{\frac{1}{\mu}} \mathcal{V}^j \cdot \nabla_Y \mathcal{V}^l.
\end{aligned}
$$

We split the nonlinear term into an oscillatory part and a mean part:

$$
\overline{Q}^k = \int_\lambda Q^k, \quad \tilde{Q}^k = Q^k - \overline{Q}^k.
$$

Note that

$$
\overline{Q}^k = Q^{k,2} + \overline{Q}^{k,3}, \quad \tilde{Q}^k = Q^{k,1} + \tilde{Q}^{k,3}.
$$

Thanks to this decomposition, we shall split (22) into the two systems

$$
\begin{cases} L^\varepsilon \mathbf{v}^k = \overline{Q}^k, \\ \nabla \cdot \mathbf{v}^k = 0, \end{cases} \tag{45}
$$

$$
\begin{cases} L^\varepsilon \mathcal{V}^k = \tilde{Q}^k - R^{k-1} + \varepsilon^N R^k, \\ \nabla \cdot \mathcal{V}^k = -d^{k-1} + \varepsilon^N d^k. \end{cases} \tag{46}
$$

Note that we shall solve exactly system (45), as we only solve approximately (46) by requiring that d^k and R^k are uniformly bounded on $[0, T_\varepsilon]$. To solve (46), we look as in (27) for a WKB expansion of $\mathcal{V}^k$ under the form

$$
\mathcal{V}^k(t, Y, \lambda) = \sum_{l=0}^{\mu(N+1)} \varepsilon^{\frac{l}{\mu}} \mathcal{V}^{k,l}(t, Y, \lambda), \tag{47}
$$

where $\mathcal{V}^{k,l} = (\mathcal{U}^{k,l}, \varepsilon \mathcal{P}^{k,l})$. Moreover, $\mathcal{V}^{k,l}$ is made of a finite number of Fourier modes:

$$
\mathcal{V}^{k,l}(t, Y, \lambda) = \sum_{|j| \leqq k+1,\, j \neq 0} a^{k,l,j}(t, Y) \mathrm{e}^{i j \lambda} + \text{c.c.} \tag{48}
$$

Note that, as we shall solve (45) exactly, it is legitimate to have residual terms R^k which have zero mean. In particular, by the construction in Section 2.2, we already have that $\int_\lambda R^0 = 0$. By the choice of the expansion (47) and (48), this property will persist.

By using the same computation as in Section 2.2, we find that $(\mathcal{V}^{k,l}, \mathcal{V}^{k,l+1} \cdot \nabla\varphi, \mathcal{P}^{k,l})$ solves the system

$$\partial_\lambda \mathcal{V}^{k,l+1} \cdot \nabla\varphi = -\delta_{l=\mu-1} d^{k-1} + D^{k,l}, \tag{49}$$

$$|\nabla\varphi|^2\, \partial_\lambda \mathcal{P}^{k,l} = 2A\mathcal{V}^{k,l} \cdot \nabla\varphi + L^{k,l} \cdot \nabla\varphi + \ \delta_{l=0}(-R^{k-1} + \tilde{Q}^k) \ \cdot\ \nabla\varphi, \tag{50}$$

$$\begin{aligned}\partial_t \mathcal{V}^{k,l} + AY \cdot \nabla_Y \mathcal{V}^{k,l} &= \mathcal{M}(\nabla\varphi)\mathcal{V}^{k,l} - \nu|\nabla\varphi|^2 \partial^2_{\lambda\lambda} \mathcal{V}^{k,l} \\ &\quad + \mathcal{P}(\nabla\varphi)\left(L^{k,l} + \delta_l(-R^{k-1} + \tilde{Q}^k)\right),\end{aligned} \tag{51}$$

where $L^{k,l}$ depends only on $\nabla\varphi$, $(\mathcal{V}^{k,j}, \mathcal{V}^{k,j+1}\cdot\nabla\varphi, \mathcal{P}^{k,j})_{j\leqq l-1}$ and on $(\mathcal{V}^j)_{j\leqq k-1}$. Similarly, $D^{k,l}$ depends on $\nabla\varphi$ and $(\mathcal{V}^{k,j})_{j\leqq l}$, $(\mathcal{V}^l)_{l\leqq k-1}$.

We set $\mathcal{V}^{k,l} = 0$ for $l < 0$ so that the previous expression makes sense for every $l \in \mathbb{Z}$. In the previous expressions, $\delta_{l=j}$ stands for the Kronecker symbol: $\delta_{l=j} = 1$ if $l = j$ and 0 otherwise. An important remark is that we have in particular that

$$\partial_\lambda \mathcal{V}^{k,0} \cdot \nabla\varphi = 0. \tag{52}$$

The next step is to estimate the solutions of (49)–(51) and of (45). The difficulty, which is actually one of the main difficulty of this paper, is to estimate (with a good control of the growth) the solution of (45) which is a two-dimensional linearized Navier–Stokes equation. Because of the divergence-free condition, we cannot find an expansion with profiles compactly supported (or fastly decreasing) in the vicinity of the stagnation point to reduce the problem to a simpler one as was possible for the oscillating part. This is why we shall directly estimate the exact solution. Our assumption (H2) will be crucial in this step.

2.4. Two-dimensional linearized Navier–Stokes equation

In this section we state the estimates that we shall use for the control of the solution of (45).

We consider the solution $\mathbf{v}(t, x_1, x_2) \in \mathbb{R}^3$ of the following linearized two-dimensional Navier–Stokes equation in $\mathbb{R}^2$ or $\mathbb{T}^2$ with zero initial data:

$$\partial_t \mathbf{v} + u^s \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \nabla u^s + \nabla p = \nu\varepsilon^2 \Delta\mathbf{v} + \mathcal{F}, \quad \nabla \cdot \mathbf{v} = 0 \tag{53}$$

with $\mathcal{F} = \mathcal{F}(t, x_1, x_2)$. By setting $\mathbf{v} = (v, v_3)$, $\mathcal{F} = (F, F_3)$ with $v,\ F \in \mathbb{R}^2$, we can decouple (53) into the usual two-dimensional Navier–Stokes equation for v

$$\partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + \nabla p = \nu\varepsilon^2 \Delta v + F, \quad \nabla \cdot v = 0 \tag{54}$$

and a convection–diffusion equation for v_3,

$$\partial_t v_3 + u^s \cdot \nabla v_3 - \nu\varepsilon^2 \Delta v_3 = F_3. \tag{55}$$

Note that it would be possible to perform a WKB expansion of (55) by looking for solutions under the form

$$v_3 \sim \sum_l \varepsilon^{\frac{l}{\mu}} V_3^l\left(\frac{y}{\varepsilon^\beta}\right)$$

with profiles compactly supported in Y if the source term F_3 is compactly supported. Nevertheless, such a WKB analysis is not possible for the two-dimensional Navier–Stokes equation (54) because of the nonlocal constraint $\nabla \cdot v = 0$. To overcome this difficulty, we shall directly deal with the exact solution of (54) and estimate it in a precise way. In order to have a unified approach we use the same technique for the convection diffusion equation (55) even if it is not necessary.

Before stating the lemma, we define a family of weighted norms

$$N^\varepsilon(f) = \varepsilon^{\frac{-1}{\mu}}(\|f\|_{L^1} + \|f\|) + \|f\|_{L^{\frac{2\mu-2}{\mu-2}}}, \tag{56}$$

$$Y^\varepsilon(f) = \varepsilon^{\frac{-1}{\mu}}\|f\| + \|f\|_{L^\infty} \tag{57}$$

We recall that $\|\cdot\|$ always stands for the L^2 norm. Moreover, when N^ε, Y^ε are applied to a vector this means the sum of the norm of all the components. We also define some weighted norms which control higher-order derivatives:

$$N^\varepsilon_{\mathrm{glob},m}(f) = \sum_{|\alpha|\leqq m} \varepsilon^{|\alpha|} N^\varepsilon(\partial^\alpha f),$$
$$Y^\varepsilon_{\mathrm{glob},m}(f) = \sum_{|\alpha|\leqq m} \varepsilon^{|\alpha|} Y^\varepsilon(\partial^\alpha f).$$

Note that the norms $N^\varepsilon_{\mathrm{glob},m}$, $Y^\varepsilon_{\mathrm{glob},m}$ give a control of m derivatives.

We have the following lemma

Lemma 4. (Global estimates) *Assume that the source term $F \in X$ enjoys for some $\gamma \geqq 2\sigma$ the estimate*

$$\|F\| + \|H\| \leqq C\varepsilon^{\frac{1}{\mu}} \mathrm{e}^{\gamma t} \tag{58}$$

with $H = \mathrm{curl} F = \partial_1 F_2 - \partial_2 F_1$ together with the global estimates

$$N^\varepsilon_{\mathrm{glob},m}(H) + \varepsilon^{-\beta} N^\varepsilon_{\mathrm{glob},m}(F_3) \leqq C_m \mathrm{e}^{\gamma t} \tag{59}$$

then under the assumption (H2), *we have for the vorticity $\omega = \mathrm{curl} v$, the global estimates*

$$N^\varepsilon_{\mathrm{glob},m}(\omega) + \varepsilon N^\varepsilon_{\mathrm{glob},m}(\nabla\omega) \leqq C_m \mathrm{e}^{\gamma t}. \tag{60}$$

Moreover, the velocity field $\mathbf{v}$ *enjoys the estimates*

$$Y^\varepsilon_{\mathrm{glob},m}(\mathbf{v}) \leqq C_m \mathrm{e}^{\gamma t} \tag{61}$$

$$\varepsilon^{-\beta} N^\varepsilon_{\mathrm{glob},m}(v_3) + \varepsilon\varepsilon^{-\beta} N^\varepsilon_{\mathrm{glob},m}(\nabla v_3) \leqq C_m \mathrm{e}^{\gamma t}. \tag{62}$$

The main result of this lemma is that the ε-weighted derivatives of the vorticity keep the same growth rate as the source term (namely γ). The small but nonzero viscosity is crucial to get this kind of estimates. Indeed, such estimates are in general false for the linearized Euler equation: the derivatives have a faster growth when there is an hyperbolic stagnation point in u^s.

Though very useful, the estimates of Lemma 4 are not sufficient for our purpose because only $\varepsilon^{|\alpha|}\partial^\alpha$ is controlled whereas to be in agreement with the WKB expansion (44), we would like $\varepsilon^{\beta|\alpha|}\partial^\alpha$ to be controlled. We shall see that it is indeed possible to get these kind of estimates locally in the vicinity of the elliptic stagnation point.

We define a family of truncation functions $(\chi_k)_{1\leqq k\leqq K}$ smooth and compactly supported in $\mathbb{R}^2$ such that $\chi_k \in [0, 1]$, $\chi_k = 1$ on the support of χ [which was defined in (41)] and

$$\chi_k \chi_{k-1} = \chi_k. \tag{63}$$

In words, the value of χ_{k-1} is one on the support of χ_k. An important consequence of this property is that

$$|\nabla^m \chi_k| \leqq C_m \chi_{k-1} \tag{64}$$

for every m. Finally, we set

$$\chi_k^\varepsilon(y) = \chi_k\left(\frac{y}{\varepsilon^\beta}\right). \tag{65}$$

By using this family of truncation functions, we define the weighted local norms:

$$N^\varepsilon_{k,\mathrm{loc},m}(f) = \sum_{2|\alpha|+|\alpha'|\leqq m} \varepsilon^{\beta|\alpha|+|\alpha'|} N^\varepsilon\big(\chi^\varepsilon_{k+|\alpha|}\partial^{\alpha+\alpha'} f\big),$$
$$Y^\varepsilon_{k,\mathrm{loc},m}(f) = \sum_{2|\alpha|+|\alpha'|\leqq m} \varepsilon^{\beta|\alpha|+|\alpha'|} Y^\varepsilon\big(\chi^\varepsilon_{k+|\alpha|}\partial^{\alpha+\alpha'} f\big).$$

Note that the norms $N^\varepsilon_{k,\mathrm{loc},m}$, $Y^\varepsilon_{k,\mathrm{loc},m}$ also give a control of m derivatives but that $m/2$ of them have a better local behavior in the sense that $\varepsilon^{\beta m/2}\nabla^{m/2}$ is controlled and not $\varepsilon^{m/2}\nabla^{m/2}$.

We can prove the following local estimates:

Lemma 5. (Local estimates) *Under the same assumptions as in Lemma* 4, *if moreover, the source term enjoys the local estimates*

$$N^\varepsilon_{k,\mathrm{loc},m}(H) + \varepsilon^{-\beta} N^\varepsilon_{k,\mathrm{loc},m}(F_3) \leqq C_m \mathrm{e}^{\gamma t} \tag{66}$$

for k, m such that $k + m \leqq K$, then we also have for ω the local estimates

$$N^\varepsilon_{k,\mathrm{loc},m}(\omega) + \varepsilon N^\varepsilon_{k,\mathrm{loc},m}(\nabla\omega) \leqq C_m \mathrm{e}^{\gamma t}. \tag{67}$$

Moreover, the velocity field **v** *also enjoys the local estimates*

$$Y^\varepsilon_{k,\mathrm{loc},m}(\mathbf{v}) \leqq C_m \mathrm{e}^{\gamma t}, \tag{68}$$

$$\varepsilon^{-\beta} N^\varepsilon_{k,\mathrm{loc},m}(v_3) + \varepsilon\varepsilon^{-\beta} N^\varepsilon_{k,\mathrm{loc},m}(\nabla v_3) \leqq C_m \mathrm{e}^{\gamma t}. \tag{69}$$

The main result of this lemma is that locally in the vicinity of the stagnation point, we can have a control on $\varepsilon^{\beta|\alpha|}\partial^\alpha$ as suggested by the WKB expansion (44).

The proof of Lemmas 4 and 5 is postponed until Section 2.7. We shall first explain how to end the proof of Theorem 1 by using these estimates.

2.5. Estimates of U^k, $k \geqq 1$

We shall now use the previous results to derive precise estimates on U^k. Note that in our argument, M (the number of terms in (20)) and K (defined in Section 2.4) are free parameters. They will be chosen carefully later. We define a sequence m_k such that

$$m_{k+1} = \text{Min}\left(\frac{m_k}{2} - 1, m_k - (\mu(N+1)+3)\right)$$

with $m_1 = K$.

The main result of this section is

Lemma 6. *Under the assumptions of Theorem* 1*, we have for every k such that $k + m_k \leqq K$ the uniform estimates on* $[0, T^\varepsilon]$*:*

$$|\partial_Y^\alpha \mathcal{U}^k(t, Y, \lambda)| \leqq C_{m_k} \mathrm{e}^{(k+1)\sigma t} \mathbf{1}_{D(0,1)} \tag{70}$$

$$\left|\partial_Y^\alpha \left(\mathcal{V}^k \cdot \nabla_X \varphi\right)\right| \leqq C_{m_k} \varepsilon^{\frac{1}{\mu}} \mathrm{e}^{(k+1)\sigma t}, \tag{71}$$

$$|\partial_Y^\alpha R^k| + |\partial_Y^\alpha d^k| \leqq C_{m_k} \mathrm{e}^{\sigma(k+1)t} \tag{72}$$

for $|\alpha| \leqq m_k$. Moreover, $\mathbf{v}^k$ *and* $\omega^k := \mathrm{curl} v^k$ *satisfy the global and local estimates*

$$N^\varepsilon_{\mathrm{glob},m}(\omega^k) + Y^\varepsilon_{\mathrm{glob},m}(\mathbf{v}^k) + \varepsilon^{-\beta} N^\varepsilon_{\mathrm{glob},m}(v_3^k) \leqq C_{m_k} \mathrm{e}^{(k+1)\sigma t}, \tag{73}$$

$$N^\varepsilon_{k,\mathrm{loc},m}(\omega^k) + Y^\varepsilon_{k,\mathrm{loc},m}(\mathbf{v}^k) + \varepsilon^{-\beta} N^\varepsilon_{k,\mathrm{loc},m}(v_3^k) \leqq C_{m_k} \mathrm{e}^{(k+1)\sigma t} \tag{74}$$

for $m \leqq m_k$.

Proof of Lemma 6. We perform an induction over k.

For $k = 1$, We have to solve (45) and (46). Note that in this case $Q^{1,2} = 0$ so that the source term depends only on $\mathcal{U}^0$, which is estimated in Lemma 3. The resolution of (46) by using the expansion (47) and (48) reduces to the study of (49)–(51). The study of this system is easy. Indeed by using Lagrangian coordinates as in (39), the equation of the amplitudes $a^{1,l,j}$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} a^{1,l,j}(t, \mathrm{e}^{tA} Y) = \left(\mathcal{M}(\xi(t) - \nu j^2 |\xi(t)|^2\right) a^{k,l,j} + \mathcal{O}(\mathrm{e}^{2\sigma t}).$$

Consequently, thanks the choice of σ, a standard argument of ODE gives

$$a^{1,l,j} = \mathcal{O}(\mathrm{e}^{2\sigma t}).$$

The estimates (70)–(72) follow easily by the same method as in the proof of Lemma 3; for $k = 1$, they actually hold for every α.

It remains to study (45). Of course, we shall use the estimates of Lemmas 4 and 5. Again, for $k = 1$, we notice that $\overline{Q}^k$ depends only on $\mathcal{V}^0$, we have

$$\overline{Q}^1 = \overline{Q}^{1,3} = \int_\lambda \mathcal{V}^0 \cdot \nabla_X \varphi \mathcal{V}^0 + \varepsilon^{\frac{1}{\mu}} \mathcal{V}^0 \cdot \nabla_Y \mathcal{V}^0.$$

Note that $\overline{Q}^1$ is compactly supported in $B^\varepsilon := B(0, \varepsilon^\beta)$ so that the local estimates imply the global ones. By using the estimates of Lemma 3 and especially the improved estimates (36) on the divergence, we get for every α

$$\varepsilon^{\beta|\alpha|} |\partial_y^\alpha \overline{Q}^1| \leqq C_\alpha \varepsilon^{\frac{1}{\mu}} \mathrm{e}^{2\sigma t} \mathbf{1}_{B^\varepsilon}. \tag{75}$$

In particular, this yields for every α,

$$N^\varepsilon_{1,\mathrm{loc},m}(\mathrm{curl}\, \overline{Q}^1_h) + \varepsilon^{-\beta} N^\varepsilon_{1,\mathrm{loc},m}(\overline{Q}^1) \leqq C_m \mathrm{e}^{2\sigma t},$$

where Q_h stands for the horizontal part of Q, that is, $Q_h = (Q_1, Q_2)$. Therefore the assumptions of Lemmas 4 and 5 are satisfied and the conclusion of these lemmas holds for $m \leqq m_1 \leqq K$. This gives (73) and (74) for $k = 1$.

Now, let us assume that the estimates of Lemma 6 are true for $j \leqq k - 1$.

At first, we shall solve (46). Note that by the induction assumption, we have

$$|\partial_Y^\alpha R^{k-1}| \leqq C_{m_{k-1}} \mathrm{e}^{k\sigma t}, \quad |\alpha| \leqq m_{k-1}. \tag{76}$$

Next, we use again the fact that $\mathcal{V}^j$ are compactly supported in B^ε and the induction assumption, to write

$$\begin{aligned}
|\partial_Y^\alpha Q^{k,1}| &= |\varepsilon^{\beta|\alpha|} \partial_y^\alpha Q^{k,1}| \\
&\leqq C \sum_{j+l=k-1} \|\mathcal{V}^j\|_{W^{|\alpha|+1,\infty}} \Big(\varepsilon^{\beta|\alpha|} \|\mathbf{1}_{B^\varepsilon} v^l\|_{W^{|\alpha|,\infty}} \\
&\quad + \varepsilon^{\beta|\alpha|+1} \|\mathbf{1}_{B^\varepsilon} \nabla v^l\|_{W^{|\alpha|,\infty}} \Big) \\
&\leqq C \sum_{j+l=k-1} \|\mathcal{V}^j\|_{W^{|\alpha|+1,\infty}} Y^\varepsilon_{l,\mathrm{loc},2(|\alpha|+1)}(v^l) \\
&\leqq C \mathrm{e}^{(k+1)\sigma t}
\end{aligned} \tag{77}$$

for α such that $2(|\alpha| + 1) \leqq m_{k-1}$. In a similar way, $\tilde{Q}^{k,3}$ is compactly supported in B^ε and we have

$$|\partial_Y^\alpha \tilde{Q}^{k,3}| \leqq C \mathrm{e}^{(k+1)\sigma t} \tag{78}$$

for $|\alpha| \leqq m_{k-1} - 1$. Now that (76)–(78) are established, we can solve (46) by using the expansion (47). As previously, this part of the analysis reduces to the study of linear ordinary differential equations. By repeating the proof of Lemma 3, it is easy to see that $\partial^\alpha \mathcal{V}^k$ is estimated with respect to $\partial^{\alpha+B}(R^{k-1} + Q^{k,1} + \tilde{Q}^{k,3})$, where B is a fixed number of derivatives, we can take $B = \mu(N + 1) + 2$. Consequently, we easily obtain (70)–(72) when $|\alpha| + B + 1 \leqq m_{k-1}$, that is, when $|\alpha| \leqq m_k$ by definition of m_k.

It remains to study (45). We want to apply Lemmas 4 and 5, so we have to estimate the source term. Since $\overline{Q}^{k,3}$ is compactly supported in B^ε, we only need to perform the local estimates. By using the induction assumption, we get thanks to (31)

$$N^\varepsilon_{k,\mathrm{loc},m}(\mathrm{curl}\,\overline{Q}^{k,3}) \lesssim \mathrm{e}^{\sigma(k+1)t}, \tag{79}$$

$$N^\varepsilon_{k,\mathrm{loc},m}(\overline{Q}^{k,3}) \lesssim \mathrm{e}^{\sigma(k+1)t} \tag{80}$$

for $m \leqq m_k$ as long as $m_k + 2 \leqq m_{k-1}$. We now turn to the estimate of $Q^{k,2}$. Note that

$$Q^{k,2} = \varepsilon \sum_{j+l \leqq k-1} \mathbf{v}^j \cdot \nabla \mathbf{v}^l = \varepsilon \sum_{j+l \leqq k-1} v^j \cdot \nabla_y \mathbf{v}^l$$

since v^l does not depend on x_3. The L^2 estimate is straightforward: we have

$$\begin{aligned} \|Q^{k,2}\| &\lesssim \sum_{j+l \leqq k-1} \|v^j\|_{L^\infty}\, \varepsilon \|\nabla \mathbf{v}^l\| \\ &\lesssim \sum_{j+l \leqq k-1} Y^\varepsilon_{\mathrm{glob},0}(v^j)\, \varepsilon^{\frac{1}{\mu}}\, Y^\varepsilon_{\mathrm{glob},1}(\mathbf{v}^l) \\ &\lesssim \varepsilon^{\frac{1}{\mu}} \mathrm{e}^{\sigma(k+1)t}. \end{aligned} \tag{81}$$

Now let us set $H^k = \partial_1 Q_2^{k,2} - \partial_2 Q_1^{k,2}$ as in Lemma 4. Towards the estimate of H^k, we rewrite $Q_h^{k,2}$ in a more-symmetric form

$$Q_h^{k,2} = \frac{\varepsilon}{2} \sum_{j+l \leqq k-1} v^j \cdot \nabla v^l + v^l \cdot \nabla v^j \tag{82}$$

and hence, since $\nabla \cdot v^i = 0$, for $i \leqq k-1$, we get

$$H^k = \frac{\varepsilon}{2} \sum_{j+l \leqq k-1} v^j \cdot \nabla \omega^l + v^l \cdot \nabla \omega^j. \tag{83}$$

This yields

$$\begin{aligned} \|H^k\| &\lesssim \sum_{j+l \leqq k-1} \|v^j\|_{L^\infty}\, \varepsilon \|\nabla \omega^l\| \\ &\lesssim \varepsilon^{\frac{1}{\mu}} \sum_{j+l \leqq k-1} Y^\varepsilon_{\mathrm{glob},0}(v^j)\, N^\varepsilon_{\mathrm{glob},1}(\omega^l) \lesssim \varepsilon^{\frac{1}{\mu}} \mathrm{e}^{\sigma(k+1)t}. \end{aligned} \tag{84}$$

More generally, we obtain

$$N^\varepsilon_{\mathrm{glob},m_k}(H^k) \lesssim \sum_{j+l \leqq k-1} Y^\varepsilon_{\mathrm{glob},m_k}(v^j) N^\varepsilon_{\mathrm{glob},m_k+1}(\omega^l) \lesssim \mathrm{e}^{\sigma(k+1)t}, \tag{85}$$

if $m_k + 1 \leqq m_{k-1}$.

In a similar way, we get

$$N^{\varepsilon}_{\mathrm{glob},m_k}(\overline{Q}_3^{k,2}) \lesssim \sum_{j+l\leqq k-1} Y^{\varepsilon}_{\mathrm{glob},m_k}(v^j) N^{\varepsilon}_{\mathrm{glob},m_k+1}(v_3^l) \lesssim \varepsilon^{\beta} \mathrm{e}^{\sigma(k+1)t}. \quad (86)$$

For the local estimates, we use again a consequence of the property (63), which is

$$\chi_{k+|\alpha|} \leqq C\, \chi_{k-1+|\alpha|}.$$

We find

$$\begin{aligned} N^{\varepsilon}_{k,\mathrm{loc},m}(H^k) &\lesssim \sum_{j+l\leqq k-1} Y^{\varepsilon}_{k-1,\mathrm{loc},m}(v^j) N^{\varepsilon}_{k-1,\mathrm{loc},m+1}(\omega^l) \\ &\lesssim \mathrm{e}^{\sigma(k+1)t} \end{aligned} \quad (87)$$

and

$$N^{\varepsilon}_{k,\mathrm{loc},m}(\overline{Q}_3^{k,2}) \lesssim \varepsilon^{\beta} \mathrm{e}^{\sigma(k+1)t}, \quad (88)$$

if $m+1 \leqq m_{k-1}$ and $k+m \leqq K$. Finally, thanks to (79), (81), (84)–(88) all the assumptions of Lemmas 4 and 5 are matched and hence we can use these lemmas to get (73) and (74). This ends the proof of Lemma 6.

2.6. End of the Proof of Theorem 1

The end of the proof uses the argument of [6] and Lemma 6. We choose M such that $M\sigma > \|\nabla u^s\|_{L^\infty} + 2$. Then, we choose K sufficiently large such that $m_M \geqq 3$. Let us define

$$\mathrm{NS}(U) = \partial_t v + u^s \cdot \nabla v + v \cdot \nabla u^s + v \cdot \nabla v + \nabla p - \nu \varepsilon^2 \Delta v.$$

Then, by the choice of the terms of (20), U^{app} is a solution of

$$\mathrm{NS}(U^{\mathrm{app}}) = R^{\mathrm{app}}, \quad \nabla \cdot v^{\mathrm{app}} = D^{\mathrm{app}},$$

where

$$\begin{aligned} D^{\mathrm{app}} &= \varepsilon^{N(M+1)} d^M, \\ R^{\mathrm{app}} &= \varepsilon^{N(M+1)} R^M + \varepsilon \sum_{M\leqq j+l\leqq 2M} \varepsilon^{N(j+l+2)} V^j \cdot \nabla V^l. \end{aligned}$$

By using Lemma 6, we have the estimates

$$\|\varepsilon^{|\alpha|} \partial_y^{\alpha} D^{\mathrm{app}}\| \leqq C \varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t} \quad (89)$$

thanks to (72) for $|\alpha| \leqq 3$. Moreover, we can also write

$$\|R^{\mathrm{app}}\| \leqq C\varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t} + C \sum_{M\leqq j+l\leqq 2M} \varepsilon^{N(j+l+2)} \varepsilon \|V^j \cdot \nabla V^l\|.$$

Next, we notice that we can split the term $\varepsilon V^j \cdot \nabla V^l$ into two parts. The first term is uniformly bounded and compactly supported in $D(0, \varepsilon^\beta)$ thanks to (70). Consequently, its L^2 norm is bounded by ε^β. The second term is

$$\sum_{M \leqq j+l \leqq 2M} \varepsilon^{N(j+l+2)} \varepsilon \|\mathbf{v}^j \cdot \nabla \mathbf{v}^l\| = \sum_{M \leqq j+l \leqq 2M} \varepsilon^{N(j+l+2)} \varepsilon \|v^j \cdot \nabla \mathbf{v}^l\|.$$

We estimate this term thanks to (73)

$$\begin{aligned} &\sum_{M \leqq j+l \leqq 2M} \varepsilon^{N(j+l+2)} \varepsilon \|\mathbf{v}^j \cdot \nabla \mathbf{v}^l\| \\ &\quad \leqq \varepsilon \sum_{M \leqq j+l \leqq 2M} \varepsilon^{N(j+l+2)} \|v^j\|_{L^\infty} (\|\nabla v^l| + \|\nabla \mathbf{v}_3^l\|) \\ &\quad \leqq \varepsilon \sum_{M \leqq j+l \leqq 2M} \varepsilon^{N(j+l+2)} \|v^j\|_{L^\infty} (\|\omega^l\| + \|\nabla \mathbf{v}_3^l\|) \\ &\quad \leqq \varepsilon^\beta \sum_{M \leqq k \leqq 2M} \varepsilon^{N(k+2)} \mathrm{e}^{\sigma(k+2)t}. \end{aligned}$$

Next we notice that

$$\sum_{M \leqq k \leqq 2M} \varepsilon^{N(k+2)} \mathrm{e}^{\sigma(k+2)t} \leqq C \varepsilon^{N(M+2)} \mathrm{e}^{\sigma(M+2)t}$$

for $t \in [0, T^\varepsilon - T_1]$ with T_1 sufficiently large independent of ε. Consequently, we get that

$$\begin{aligned} \|R^{\mathrm{app}}\| &\leqq C\varepsilon^N \varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t} + C\varepsilon^\beta \varepsilon^{N(M+2)} \mathrm{e}^{\sigma(M+2)t} \\ &\leqq C\varepsilon^\beta \varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t} \end{aligned} \tag{90}$$

for $t \in [0, T^\varepsilon - T_1]$. Indeed, we have used that $N > \beta$ and that $\varepsilon^N \mathrm{e}^{\sigma t} \leqq 1$.

Next, we choose in a classical way a corrector V^c such that

$$\nabla \cdot V^c = -D^{\mathrm{app}}.$$

Thanks to (89), V^c enjoys the estimates

$$\|V^c\|_{H^s} \leqq C\varepsilon^{N-s} \varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t}, \quad s \leqq 3. \tag{91}$$

This allow us to set

$$V^\varepsilon = V^{\mathrm{app}} + V^c, \quad P^\varepsilon = P^{\mathrm{app}}, \quad U^\varepsilon = (V^\varepsilon, P^\varepsilon)$$

so that

$$NS(V^\varepsilon) = R^\varepsilon, \quad \nabla \cdot V^\varepsilon = 0 \tag{92}$$

with thanks to (90), (91) for $N \geqq 3 + \beta$ the estimate

$$\|R^\varepsilon\| \leqq C\varepsilon^\beta \varepsilon^{N(M+1)} \mathrm{e}^{\sigma(M+1)t} \tag{93}$$

for $t \in [0, T^\varepsilon - T_1]$. The last step is to construct an exact solution of (18). We set $v = V^\varepsilon + w$ with w a Leray solution of

$$\partial_t w + (u^s + V^\varepsilon) \cdot \nabla w + w \cdot \nabla (u^s + V^\varepsilon) + \varepsilon w \cdot \nabla w + \nabla p - \nu \varepsilon^2 \Delta w = R^\varepsilon, \tag{94}$$

with the constraint $\nabla \cdot w = 0$ and zero initial value. The standard energy estimate for the Navier–Stokes equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w\| \leqq C \varepsilon^\beta \varepsilon^{N(m+1)} \mathrm{e}^{\sigma(M+1)t} + \left(\|\nabla u^s\|_{L^\infty} + \|\nabla V^\varepsilon\|_{L^\infty}\right) \|w\|. \tag{95}$$

Note that, thanks to Lemma 6 and (91), we have

$$\|\nabla V^\varepsilon(t)\|_{L^\infty} \leqq C \sum_{k=0}^{M} \varepsilon^{N(k+1)} \mathrm{e}^{\sigma(k+1)t}.$$

Therefore, by enlarging T_1, we can have

$$\|\nabla V^\varepsilon(t)\|_{L^\infty} \leqq 2$$

and hence, we deduce from (95) and the Gronwall inequality that

$$\|w(t)\| \leqq C\varepsilon^\beta \, \varepsilon^{N(m+1)} \mathrm{e}^{\sigma(M+1)t}. \tag{96}$$

The conclusion of Theorem 1 follows easily. Indeed, we have

$$\|v(0)\|_{H^s} = \|V^0\|_{H^s} = C\varepsilon^{N-s+\beta}$$

and

$$\begin{aligned}\|v(T)\|_{L^2(D(0,\varepsilon^\beta))} &\geqq \|\mathcal{V}^0(t)\| - C\varepsilon^\beta \sum_{k=1}^{M} \varepsilon^{N(k+1)} \mathrm{e}^{\sigma(k+1)t} \\ &\geqq \varepsilon^\beta \varepsilon^N \mathrm{e}^{\sigma T} \left(1 - C \sum_{k=1}^{M} \varepsilon^{Nk} \mathrm{e}^{M\sigma k t}\right).\end{aligned}$$

Consequently, by choosing $T = T^\varepsilon - T_1 - T_2$ with T_2 sufficiently large and independent of ε, we get by using (34)

$$\|v(T)\|_{L^2(D(0,\varepsilon^\beta))} \geqq \eta \, \varepsilon^\beta.$$

Moreover, since

$$\sup_{[0,T]} \|v(t)\|_{L^\infty(D(0,\varepsilon^\beta))} \geqq \varepsilon^{-\beta} \sup_{[0,T]} \|v(t)\|_{L^2(D(0,\varepsilon^\beta))}$$

we also get the local instability in the L^∞ norm. This ends the proof of Theorem 1.

2.7. Proof of the technical Lemmas

2.7.1. Proof of Lemma 4 We first establish the estimates (60) on the vorticity; the estimates (61) for the horizontal part v will be deduced from them. The estimates on v_3 will be shown at the end. We give the proof for the domain $\mathbb{R}^2$, the proof for $\mathbb{T}^2$ being similar and slightly easier due to the absence of low frequencies. We shall use the equation for the vorticity $\omega = \mathrm{curl} v$. We can express the vorticity with respect to the velocity thanks to the Biot–Savart law:

$$v = K \star \omega, \quad \hat{K}(\xi) = -\mathrm{i}\frac{\xi^{\perp}}{|\xi|^2}, \tag{97}$$

where $\hat{K}$ is the Fourier transform of K. We recall the following useful estimates (see [5, 24], for example)

$$1 < p < 2, \quad 2 < q < +\infty, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2}, \quad \|v\|_{L^q} \leqq C\|\omega\|_{L^p}, \tag{98}$$

$$1 \leqq p < 2, \quad 2 < q \leqq \infty, \quad \frac{1}{2} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad \|v\|_{L^\infty} \leqq C\|\omega\|_{L^p}^{\alpha}\, \|\omega\|_{L^q}^{1-\alpha}, \tag{99}$$

$$1 < p < +\infty, \quad \|\nabla v\|_{L^p} \leqq C\|\omega\|_{L^p}. \tag{100}$$

Note that, thanks to (H2), we already have

$$\|v(t)\| + \|\omega(t)\| \leqq C\varepsilon^{\frac{1}{\mu}} \mathrm{e}^{\gamma t} \quad \forall t \geqq 0. \tag{101}$$

By taking the curl of (54), we get

$$\partial_t \omega + u^s \cdot \nabla\omega + v \cdot \nabla\omega^s - \nu\varepsilon^2 \Delta\omega = H, \tag{102}$$

where we have set $H = \mathrm{curl} F$. By using the semigroup $S(t)$ of the convection–diffusion operator $\partial_t + u^s \cdot \nabla - \nu\varepsilon^2\Delta$ (which depends on $\nu\varepsilon^2$ even if we do not mention this dependence explicitly), we can rewrite the solution of (102) as

$$\omega(t, x) = \int_0^t S(t-\tau)\left(-v \cdot \nabla\omega^s + H\right)(\tau)\,\mathrm{d}\tau. \tag{103}$$

Since u^s is divergence free, $S(t, \tau)$ shares estimates with the semigroup of the heat equation. Namely, for every $p \in [1, +\infty]$ and $t > \tau > 0$,

$$\|S(t)\|_{\mathcal{L}(L^p)} \leqq 1, \quad \|\nabla S(t)\|_{\mathcal{L}(L^p)} \leqq \frac{C}{\sqrt{\nu\varepsilon^2}} \max\left(1, \frac{1}{\sqrt{t}}\right), \tag{104}$$

where $C = C\left(\|\nabla u^s\|_{L^\infty}\right)$. The first inequality in (104) follows directly from energy estimates. The second inequality is shown in the Appendix.

We now derive the estimate (60), starting with $m = 0$. Thanks to (103) and (104)

$$\begin{aligned}\|\omega\|_{L^1} &\leqq C\int_0^t ||H(\tau)\|_{L^1} + \|v\cdot\nabla\omega^s(\tau)\|_{L^1}\mathrm{d}\tau \\ &\leqq C\int_0^t \|H(\tau)\|_{L^1} + ||v(\tau)\|_{L^2}\,\mathrm{d}\tau\end{aligned}$$

and hence thanks to (101) and (59) we get

$$\|\omega\|_{L^1} \leqq C\varepsilon^{\frac{1}{\mu}}\mathrm{e}^{\gamma t}. \tag{105}$$

Next, we turn to the $L^{\frac{2\mu-2}{\mu-2}}$ estimate. By (103) and (104), we get

$$\|\omega\|_{L^{\frac{2\mu-2}{\mu-2}}} \leqq C\int_0^t \|H\|_{L^{\frac{2\mu-2}{\mu-2}}} + \|v\|_{L^\infty}\,\mathrm{d}\tau$$

and we use (99) with $p = 1, q = \frac{2\mu-2}{\mu-2}$. Thanks to the Young inequality, this yields

$$\|\omega\|_{L^{\frac{2\mu-2}{\mu-2}}} \leqq \int_0^t C\|H\|_{L^{\frac{2\mu-2}{\mu-2}}} + C(\eta)\|\omega\|_{L^1} + \eta\|\omega\|_{L^{\frac{2\mu-2}{\mu-2}}}\,\mathrm{d}\tau.$$

To conclude, we use (59), (105) and we choose $\eta < \gamma$. The Gronwall inequality yields

$$\|\omega\|_{L^{\frac{2\mu-2}{\mu-2}}} \leqq C\mathrm{e}^{\gamma t}. \tag{106}$$

Next by using again (103), (104) and (105), (106), we also get for $p \in [1, 2]$

$$\begin{aligned}\varepsilon^{\frac{-1}{\mu}}\varepsilon\|\nabla\omega||_{L^p} + \varepsilon\|\nabla\omega||_{L^{\frac{2\mu-2}{\mu-2}}} &\leqq C\int_0^t \max\left(1, \frac{1}{\sqrt{t-\tau}}\right)\mathrm{e}^{\gamma\tau}\,\mathrm{d}\tau \\ &\leqq C\mathrm{e}^{\gamma t}.\end{aligned} \tag{107}$$

Note that obtention of this last estimate requires in a crucial way the presence of the slight viscosity. The same estimate does not hold in the inviscid case. The estimates of higher-order derivatives of the vorticity follow easily by taking the derivative of the equation (102) and applying the previous method. Indeed, assume by induction that (60) holds for $|\alpha| \leqq m-1$. We can apply the operator $\varepsilon^m\partial^\alpha$ to (102) and use the Duhamel formula:

$$\varepsilon^m\partial^\alpha\omega = \int_0^t S(t-\tau)\left(\varepsilon^m\partial^\alpha H - \varepsilon^m\partial^\alpha(v\cdot\nabla\omega^s) - \varepsilon^m[\partial^\alpha, u^s\cdot\nabla]\omega\right)\,\mathrm{d}\tau. \tag{108}$$

Thanks to the second estimate of (104), we get

$$\begin{aligned}\varepsilon^{m+1}N^\varepsilon(\nabla\partial^\alpha\omega) \leqq C\int_0^t C\max\left(1, \frac{1}{\sqrt{t-\tau}}\right)\big(&\varepsilon^m N^\varepsilon(\partial^\alpha H) + \varepsilon^m N^\varepsilon(\partial^\alpha(v\cdot\nabla\omega^s)) \\ &+ \varepsilon^m N^\varepsilon([\partial^\alpha, u^s\cdot\nabla]\omega)\big)\,\mathrm{d}\tau.\end{aligned}$$

Next, repeated use of the induction assumption yields, for $p \in [1, 2]$,

$$\varepsilon^m \|\partial^\alpha (v \cdot \nabla \omega^s)\|_{L^p} \leqq C\varepsilon^m \|v\|_{H^m} \leqq C\varepsilon^m (\|v\| + \|\omega\|_{H^{m-1}}) \leqq C\varepsilon^{\frac{1}{\mu}} \mathrm{e}^{\gamma t}$$

thanks to a new use of the Cauchy–Schwarz inequality and

$$\varepsilon^m \|\partial^\alpha (v \cdot \nabla \omega^s)\|_{L^{\frac{2\mu-2}{\mu-2}}} \leqq C\varepsilon^m \|v\|_{W^{m,\infty}} \leqq C\varepsilon^m \|\omega\|_{W^{m,1}}^\alpha \|\omega\|_{W^{m,\frac{2\mu-2}{\mu-2}}}^{1-\alpha} \leqq C\mathrm{e}^{\gamma t},$$

where we have used the relation $\partial^\alpha v = K \star \partial^\alpha \omega$ and (99) in the last line. Moreover, we also have

$$\begin{aligned} \varepsilon^m N^\varepsilon([\partial^\alpha, u^s \cdot \nabla]\omega) &\leqq C\varepsilon^m \left(\varepsilon^{-1/\mu} \|\omega\|_{W^{m,1}} + \varepsilon^{-1/\mu} \|\omega\|_{W^{m,2}} + \|\omega\|_{W^{m,\frac{2\mu-2}{\mu-2}}} \right) \\ &\leqq C\mathrm{e}^{\gamma t}. \end{aligned}$$

Therefore,

$$\varepsilon^{m+1} N^\varepsilon(\nabla \partial^\alpha \omega) \leqq C \int_0^t C\mathrm{e}^{\gamma\tau} \max\left(1, \frac{1}{\sqrt{t-\tau}}\right) \mathrm{d}\tau \leqq C\mathrm{e}^{\gamma t},$$

which proves (60).

To get the estimates (61) for the horizontal component v of the velocity, we first notice that the L^2 estimate of v has already been obtained in (101). For the higher-order derivatives, we use (100):

$$\varepsilon^{|\alpha|} \|\partial^\alpha v\|_{L^2} \leqq C\varepsilon^{|\alpha|} \|\omega\|_{H^{|\alpha|-1}} \leqq C\varepsilon^{1+\frac{1}{\mu}} \mathrm{e}^{\gamma t}.$$

The L^∞ estimates remain. Again, we take advantage of (99) for $p = 1, q = \frac{2\mu-2}{\mu-2}$, and deduce

$$\varepsilon^m \|\partial^\alpha v\|_{L^\infty} \leqq \varepsilon^m \|\partial^\alpha \omega\|_{L^1}^\alpha \|\partial^\alpha \omega\|_{L^{\frac{2\mu-2}{mu-2}}}^{1-\alpha} \leqq C\,\mathrm{e}^{\gamma t}.$$

At this point, we still need to get the estimates on v_3. Since v_3 is a solution of the convection diffusion equation (55), we have the Duhamel formula

$$v_3 = \int_0^t S(t-\tau) F_3(\tau)\, \mathrm{d}\tau, \tag{109}$$

and hence the estimates (62) follow from the assumption on F_3 given by (66), and from the estimate (104). The L^∞ bound is eventually recovered through the Gagliardo–Nirenberg inequality

$$\varepsilon^m \|\partial^\alpha v_3\|_{L^\infty} \leqq \left(\varepsilon^m \|\partial^\alpha v_3\|_{L^{\frac{2\mu-2}{\mu-2}}}\right)^{1-a} \left(\varepsilon^m \|\partial^\alpha \nabla v_3\|_{L^{\frac{2\mu-2}{\mu-2}}}\right)^a, \tag{110}$$

where $a = \frac{\mu-2}{\mu-1}$. This yields, thanks to (62),

$$\varepsilon^m \|\partial^\alpha v_3\|_{L^\infty} \ \leqq\ C\varepsilon^\beta\, \varepsilon^{-a} \mathrm{e}^{\gamma t}.$$

As

$$\beta - a = 1 - \frac{1}{\mu} - \frac{\mu-2}{\mu-1} = \frac{1}{\mu(\mu-1)} \ >\ 0$$

we deduce (61), which ends the proof of global estimates.

2.7.2. Proof of Lemma 5 We now turn to the proof of local estimates (68). To get the result, we shall use in the proof a slightly different family of truncation functions which depend on time. We define

$$\kappa_k(t, y) = \chi_k(\mathrm{e}^{-tA} y), \quad \kappa_k^\varepsilon = \kappa_k\left(t, \frac{y}{\varepsilon^\beta}\right).$$

Note that (63) is still true for this new family. Since there exists $c > 0$, $C > 0$ such that:

$$\forall y, \quad c|y| \leqq |\mathrm{e}^{tA} y| \leqq C|y|, \tag{111}$$

it is equivalent to prove (68) with the family χ_k^ε or with the family κ_k^ε. The main interest of our choice of this new family is that we have

$$(\partial_t + Ay \cdot \nabla)\kappa_k = 0. \tag{112}$$

To have the local estimate (68) which is better than the global ones, we shall use the special geometry of the streamlines in the vicinity of the stagnation point. We shall use the notation $u^s = Ay + v^s$.

Like in the previous lemma, we first establish (67) by induction on m. We will then focus on the estimate (68) for the horizontal velocity, and end with the treatment of v_3. Note that for $m = 0, 1$, the estimates (67) are a simple consequence of Lemma 4. For clarity of exposure, we detail the case $m = 2$, for which we have to consider two subcases: $\alpha = 0$, $|\alpha|' \leqq 2$ and $\alpha = 1$, $\alpha' = 0$. As the former one follows from Lemma 4, we only treat the latter.

Thus, let us denote $\Omega^1 = \nabla\omega$. Applying ∇ to (102) yields:

$$\left(\partial_t + u^s \cdot \nabla - \nu\varepsilon^2 \Delta\right) \Omega^1 = \nabla H - A^* \Omega^1 - \nabla v^s \cdot \nabla\omega - \nabla(v \cdot \nabla\omega^s).$$

Thanks to (111), setting

$$W^1 = \mathrm{e}^{-tA^*} \Omega^1,$$

it is equivalent to estimate $\varepsilon^\beta \kappa_{k+1}^\varepsilon \nabla\omega$ or $\varepsilon^\beta \kappa_{k+1}^\varepsilon W^1$. This last function satisfies

$$\begin{aligned}
&\left(\partial_t + u^s \cdot \nabla - \nu\varepsilon^2 \Delta\right) (\varepsilon^\beta \kappa_{k+1}^\varepsilon W^1) \\
&\quad = \varepsilon^\beta \kappa_{k+1}^\varepsilon \mathrm{e}^{-tA^*} \nabla H - \varepsilon^\beta \kappa_{k+1}^\varepsilon \mathrm{e}^{-tA^*} \left(\nabla v^s \cdot \nabla\omega\right) - \varepsilon^\beta \kappa_{k+1}^\varepsilon \mathrm{e}^{-tA^*} \nabla(v \cdot \nabla\omega^s) \\
&\qquad - \varepsilon^\beta v^s \cdot \nabla\kappa_{k+1}^\varepsilon \, \mathrm{e}^{-tA^*} \nabla\omega + \nu\varepsilon^{2+\beta} \left[\Delta, \kappa_{k+1}^\varepsilon\right] \mathrm{e}^{-tA^*} \nabla\omega := F
\end{aligned}$$

thanks to (112), so that we can use components by components the estimates (104) for the convection–diffusion operator: for every $p \in [1, +\infty[$

$$\|W^1(t)\|_{L^p} \leqq C \int_0^t \|F(\tau)\|_{L^p} \, \mathrm{d}\tau. \tag{113}$$

It remains to bound the source term. Note that

$$\varepsilon^\beta |v^s \cdot \nabla\kappa_{k+1}^\varepsilon| \leqq C\varepsilon^{2\beta} \leqq C\varepsilon, \quad \varepsilon^\beta |\kappa_{k+1}^\varepsilon \partial_i v^s| \leqq C\varepsilon^{2\beta} \leqq C\varepsilon$$

since $2\beta > 1$. Hence, thanks to (66), we get that for $p \in [1, 2]$

$$\|F(t)\|_{L^p} \leqq C\left(\varepsilon^{\frac{1}{\mu}} e^{\gamma t} + \varepsilon \|\nabla\omega(t)\|_{L^p} + \varepsilon^2 \|\nabla^2\omega(t)\|_{L^p} + \varepsilon^\beta \|v\|_{L^2} + \varepsilon^\beta \|\nabla v\|_{L^p}\right).$$

Note that we have used again the fact that thanks to the Cauchy–Schwarz inequality

$$\|v \cdot \nabla\omega^s\|_{L^p} \leqq C\|v\|_{L^2}, \quad p \in [1, 2]. \tag{114}$$

Next, we can use (100) and the global estimates (60), (61) already obtained to get

$$\|F(t)\|_{L^p} \leqq C\varepsilon^{\frac{1}{\mu}} e^{\gamma t}, \quad p \in [1, 2]. \tag{115}$$

This finally yields, thanks to (113),

$$\|W^1(t)\|_{L^p} \leqq C\,\varepsilon^{\frac{1}{\mu}} e^{\gamma t}, \quad p \in [1, 2]. \tag{116}$$

Similarly,

$$\begin{aligned}\|F(t)\|_{L^{\frac{2\mu-2}{\mu-2}}} &\leqq C\left(e^{\gamma t} + \varepsilon\|\nabla\omega(t)\|_{L^{\frac{2\mu-2}{\mu-2}}} + \varepsilon^2\|\nabla^2\omega(t)\|_{L^{\frac{2\mu-2}{\mu-2}}} + \varepsilon^\beta\|v\|_{L^\infty} + \varepsilon^\beta\|\nabla v\|_{L^{\frac{2\mu-2}{\mu-2}}}\right)\\ &\leqq Ce^{\gamma t}.\end{aligned}$$

To get the last line, we have used again the global estimates (60), (61) which are already known. Hence, we derive the bound

$$\|W^1(t)\|_{L^{\frac{2\mu-2}{\mu-2}}} \leqq C\,e^{\gamma t}. \tag{117}$$

Moreover, by using again the second estimate of (104), we also have that

$$\begin{aligned}&\varepsilon^{\frac{-1}{\mu}}\varepsilon\|\nabla W^1(t)\|_{L^p} + \varepsilon\|\nabla W^1(t)\|_{L^{\frac{2\mu-2}{\mu-2}}}\\ &\leqq \int_0^t C\max\left(1, \frac{1}{\sqrt{t-\tau}}\right) e^{\gamma\tau}\,d\tau\\ &\leqq Ce^{\gamma t}.\end{aligned} \tag{118}$$

We now detail the estimates of higher-order derivatives. We assume that (67) is true for $m-1$. We shall prove (67) at order m. First we notice that, if $\alpha = 0$, (67) follows from Lemma 4 so that we may assume $|\alpha| \geqq 1$. With previous notations, we define inductively

$$\Omega^j = \begin{pmatrix}\nabla W_1^{j-1}\\ \vdots\\ \nabla W_{2^{j-1}}^{j-1}\end{pmatrix}, \quad W^j = e^{-tA_j}\Omega^j,$$

where $A_1 = A^*$ and $A_j = I_{2^{j-1}} \otimes A^*$. We easily see that W^j solves the equation

$$\left(\partial_t + u^s \cdot \nabla - \nu\varepsilon^2 \Delta\right) W^j = F^j \tag{119}$$

with F^j under the form

$$F^j = H^j + \mathcal{C}^{j,1} + \mathcal{C}^{j,2},$$

where H^j is the part coming from the source term and $\mathcal{C}^{j,i}$ are commutators that read:

$$\mathcal{C}^{j,1} = \sum_{l+l'=j} \star_{l,l'}(t)\, \nabla^{l+1} v^s \cdot \nabla^{l'} \omega, \quad \mathcal{C}^{j,2} = \star(t)\, \nabla^j (v \cdot \nabla \omega^s), \tag{120}$$

where $\star_{l,l'}(t)$ and $\star(t)$ are harmless matrix coefficients that are uniformly bounded in time. Finally, we define for $2|\alpha| + |\alpha'| \leqq m$

$$W^{\alpha,\alpha'} = \varepsilon^{\beta|\alpha|+|\alpha'|} \kappa^{\varepsilon}_{k+|\alpha|} \partial^{\alpha'} W^{|\alpha|}.$$

Again, thanks to (111), this is equivalent to estimate $N^{\varepsilon}_{k,\mathrm{loc},m}(\omega)$ or $\sum_{2|\alpha|+|\alpha'|\leqq m} N^{\varepsilon}(W^{\alpha,\alpha'})$. By applying $\varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^{\varepsilon}_{k+|\alpha|}\partial^{\alpha'}$ to (119), we get

$$\left(\partial_t + u^s \cdot \nabla - \nu\varepsilon^2 \Delta\right) W^{\alpha,\alpha'} = H^{\alpha,\alpha'} + \mathcal{C}^{\alpha,\alpha'}, \tag{121}$$

where $H^{\alpha,\alpha'}$ contains the derivatives of the source term, so that the assumption (66) implies

$$N^{\varepsilon}(\partial^{\alpha'} H^{\alpha,\alpha'}) \leqq C \mathrm{e}^{\gamma t}. \tag{122}$$

The term $\mathcal{C}^{\alpha,\alpha'}$ contains commutators which we write

$$\mathcal{C}^{\alpha,\alpha'} = \mathcal{C}^{\alpha,\alpha'}_1 + \mathcal{C}^{\alpha,\alpha'}_2 + \mathcal{C}^{\alpha,\alpha'}_3 + \mathcal{C}^{\alpha,\alpha'}_4 + \mathcal{C}^{\alpha,\alpha'}_5 \tag{123}$$

with

$$\mathcal{C}^{\alpha,\alpha'}_1 = \sum_{l+l'=|\alpha|} \star_{l,l'}(t)\, \varepsilon^{\beta|\alpha|+|\alpha'|}\, \kappa^{\varepsilon}_{k+|\alpha|} \nabla^{|\alpha'|} \left(\nabla^{l+1} v^s \cdot \nabla^{l'} \omega\right) \tag{124}$$

$$\mathcal{C}^{\alpha,\alpha'}_2 = \star(t)\, \varepsilon^{\beta|\alpha|+|\alpha'|}\, \kappa^{\varepsilon}_{k+|\alpha|}\, \nabla^{|\alpha|+|\alpha'|} \left(v \cdot \nabla \omega^s\right), \tag{125}$$

$$\mathcal{C}^{\alpha,\alpha'}_3 = \varepsilon^2 \left[\kappa^{\varepsilon}_{k+|\alpha|}, \Delta\right] \varepsilon^{\beta|\alpha|+|\alpha'|} \partial^{\alpha'} W^{|\alpha|}, \tag{126}$$

$$\mathcal{C}^{\alpha,\alpha'}_4 = \varepsilon^{\beta|\alpha|+|\alpha'|} \left[v^s \cdot \nabla, \kappa^{\varepsilon}_{k+|\alpha|}\right] \partial^{\alpha'} W^{|\alpha|}, \tag{127}$$

$$\mathcal{C}^{\alpha,\alpha'}_5 = \varepsilon^{\beta|\alpha|+|\alpha'|} \kappa^{\varepsilon}_{k+|\alpha|} \left[u^s \cdot \nabla, \partial^{\alpha'}\right] W^{|\alpha|}. \tag{128}$$

By using (103) component by component, we get

$$W^{\alpha,\alpha'} = \int_0^t S(t-\tau) \left(H^{\alpha,\alpha'} + \mathcal{C}^{\alpha,\alpha'}\right) \mathrm{d}\tau. \tag{129}$$

Note that we have used in a crucial way the fact that $\kappa^\varepsilon_{k+|\alpha|}$ solves (112) to get $v^s\cdot\nabla$ and not $u^s\cdot\nabla$ in the expression of $\mathcal{C}_4^{\alpha,\alpha'}$. Next, we need to study the commutators. We start with $\mathcal{C}_1^{\alpha,\alpha'}$. Thanks to the Leibnitz formula and the fact that $\nabla v^s = \mathcal{O}(\varepsilon^\beta)$ on the support of $\kappa^\varepsilon_{k+|\alpha|}$, we can write for all $l+l'=|\alpha|$,

$$\begin{aligned}
&\varepsilon^{\beta|\alpha|+|\alpha'|}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|}\nabla^{|\alpha'|}\left(\nabla^{l+1}v^s\cdot\nabla^{l'}\omega\right)\right)\\
&\quad\lesssim \varepsilon^{\beta(|\alpha|+1)+|\alpha'|}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|}\nabla^{|\alpha|+|\alpha'|}\omega\right)\\
&\qquad+\sum_{l''\leqq|\alpha|-1+|\alpha'|}\varepsilon^{\beta|\alpha|+|\alpha'|}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|}\nabla^{l''}\omega\right).
\end{aligned}$$

To estimate the first term, we rewrite it as

$$\varepsilon^{2\beta}\,\varepsilon^{\beta(|\alpha|-1)}\varepsilon^{|\alpha'|}N^\varepsilon(\kappa^\varepsilon_{k+|\alpha|}\nabla(\nabla^{(|\alpha|-1)+|\alpha'|}\omega))$$

and since $2(|\alpha|-1)+(|\alpha'|+1)\leqq m-1$ and $\kappa^\varepsilon_{k+|\alpha|}\leqq\kappa^\varepsilon_{k+|\alpha|-1}$, we get that the first term is bounded by

$$\varepsilon^{2\beta-1}\varepsilon N^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega)\lesssim \mathrm{e}^{\gamma t},$$

by induction assumption and condition $2\beta>1$. By the same argument, the second term is bounded by $\varepsilon N^\varepsilon_{k,\mathrm{loc},m-1}(\omega)$ so that we get

$$N^\varepsilon(\mathcal{C}_1^{\alpha,\alpha'})\leqq C\mathrm{e}^{\gamma t}. \tag{130}$$

Let us turn to the estimate of $\mathcal{C}_3^{\alpha,\alpha'}$. Note that the crucial property (63) of the truncation functions implies in particular that

$$|\nabla\kappa^\varepsilon_{k+|\alpha|}|\leqq C\varepsilon^{-\beta}\chi_{k+|\alpha|-1},\quad |\nabla^2\kappa^\varepsilon_{k+|\alpha|}|\leqq C\varepsilon^{-2\beta}\chi_{k+|\alpha|-1}. \tag{131}$$

and hence we can write

$$\begin{aligned}
N^\varepsilon(\mathcal{C}_3^{\alpha,\alpha'})&\lesssim \varepsilon^{\beta(|\alpha|-1)+|\alpha'|+2}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|-1}\nabla(\nabla^{|\alpha|-1+|\alpha'|+1})\right)\\
&\quad+\varepsilon^{\beta(|\alpha|-2)+|\alpha'|+2}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|-1}\nabla^{|\alpha|-1+|\alpha'|+1}\right).
\end{aligned}$$

Since $2(|\alpha|-1)+(|\alpha'|+1)\leqq m-1$, this yields

$$N^\varepsilon(\mathcal{C}_3^{\alpha,\alpha'})\leqq C\,\varepsilon N^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega)+C\,N^\varepsilon_{k,\mathrm{loc},m-1}(\omega)\leqq C\,\mathrm{e}^{\gamma t}. \tag{132}$$

To estimate $\mathcal{C}_4^{\alpha,\alpha'}$ we use again (131) and the fact that $v^s=\mathcal{O}(\varepsilon^{2\beta})$ on the support of $\chi^\varepsilon_{k+|\alpha|}$ to get

$$\begin{aligned}
N^\varepsilon(\mathcal{C}_4^{\alpha,\alpha'})&\leqq C\,\varepsilon^{\beta(|\alpha|-1)+(|\alpha'|+1)}N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|-1}\nabla(\nabla^{|\alpha|-1+|\alpha'|}\omega)\right)\\
&\leqq C\,\varepsilon N^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega)
\end{aligned}$$

and hence we find finally

$$N^\varepsilon(\mathcal{C}_4^{\alpha,\alpha'}) \leqq C\,\mathrm{e}^{\gamma t}. \tag{133}$$

Next, we notice that $\mathcal{C}_5^{\alpha,\alpha'}$ vanishes when $\alpha' = 0$ so that we only have to consider the case where $|\alpha'| > 0$. By the Leibnitz formula, we get

$$\begin{aligned} N^\varepsilon(\mathcal{C}_5^{\alpha,\alpha'}) &\lesssim \varepsilon^{\beta|\alpha|+|\alpha'|} N^\varepsilon\left(\kappa^\varepsilon_{k+|\alpha|} \nabla J^{|\alpha'|-1} \nabla^{|\alpha|}\omega\right) \\ &\lesssim \varepsilon N^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega) \lesssim \mathrm{e}^{\gamma t} \end{aligned} \tag{134}$$

We still have to handle the most difficult term, which is $\mathcal{C}_2^{\alpha,\alpha'}$. Clearly, for any finite p,

$$\|\mathcal{C}_2^{\alpha,\alpha'}\|_{L^p} \lesssim \varepsilon^{\beta|\alpha|+|\alpha'|} \|\kappa^\varepsilon_{k+|\alpha|} J^m v\|_{L^p}, \tag{135}$$

where $J^m v$ stands for the m jet of v, that is the matrix of all the derivatives of order less than m. Thus, by (63), it is enough to estimate the L^p norm of

$$\varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}\partial^{\alpha+\alpha'} v = \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|} K \star \partial^{\alpha+\alpha'}\omega, \quad |\alpha| + |\alpha'| \leqq m,\ |\alpha| \geqq 1.$$

By an integration by parts, it is equivalent to estimate terms like

$$I_m = \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}\partial_i K \star \partial^{\alpha_1+\alpha'}\omega, \quad |\alpha_1| = |\alpha| - 1,\ i = 1,\ 2.$$

We split I_m into two parts:

$$\begin{aligned} I_m &= \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}\partial_i K \star \partial^{\alpha_1}(\kappa^\varepsilon_{k+|\alpha_1|}\partial^{\alpha'}\omega) \\ &\quad + \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}\partial_i K \star \partial^{\alpha_1}((1-\kappa^\varepsilon_{k+|\alpha_1|})\partial^{\alpha'}\omega) \qquad (136) \\ &:= I_{m,1} + I_{m,2}. \qquad (137) \end{aligned}$$

The estimate of $I_{m,1}$ follows from the Calderon–Zygmund theory: by (100), we have

$$\|I_{m,1}\|_{L^p} \leqq C\varepsilon^{\beta|\alpha|+|\alpha'|}\|\partial^{\alpha_1}(\kappa^\varepsilon_{k+|\alpha_1|}\partial^{\alpha'}\omega)\|_{L^p}, \tag{138}$$

which yields by the induction assumption

$$N^\varepsilon(I_{m,1}) \lesssim \varepsilon^{\beta+\frac{1}{\mu}} N^\varepsilon_{loc,k,m-1}(\omega) \ \lesssim\ \varepsilon^{\beta+\frac{1}{\mu}}\mathrm{e}^{\gamma t}. \tag{139}$$

Note that we have used again the property (63) of the truncation functions and the induction assumption.

By definition of $I_{m,2}$, we have

$$I_{m,2} = \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}(y)\int_{y'} \partial_i K(y-y')\partial^{\alpha_1}((1-\kappa^\varepsilon_{k+|\alpha_1|})\partial^{\alpha'}\omega)(y')\,\mathrm{d}y'.$$

We notice that, for y in the support of $\kappa^\varepsilon_{k+|\alpha|}$ and y' in the support of $1-\kappa^\varepsilon_{k+|\alpha_1|}$, we have $|y-y'| \geqq c\varepsilon^\beta$ for some $c>0$ since $|\alpha_1| = |\alpha|-1$, so that $I_{2,m}$ is not a singular integral operator any more. We rewrite it as

$$I_{m,2} = \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}(y)\int_{|y'|\geqq 1} \partial_i\partial^{\alpha_1}K(y-y')(1-\kappa^\varepsilon_{k+|\alpha_1|})\partial^{\alpha'}\omega(y')\,\mathrm{d}y' \\ + \varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}(y)\int_{|y'|\leqq 1} \partial_i\partial^{\alpha_1}K(y-y')(1-\kappa^\varepsilon_{k+|\alpha_1|})\partial^{\alpha'}\omega(y')\,\mathrm{d}y'. \tag{140}$$

By homogeneity of K,

$$|\partial_i\partial^{\alpha_1}K(z)| \lesssim \frac{1}{|z|^{|\alpha_1|+2}} \lesssim \frac{1}{|z|^{|\alpha|+1}}.$$

Therefore, the L^p norm of the first integral at the right-hand side is bounded by $\varepsilon^{\beta|\alpha|+|\alpha'|}\|\partial^{\alpha'}\omega\|_{L^p}$. The second integral is controlled in the following way:

$$\varepsilon^{\beta|\alpha|+|\alpha'|}\kappa^\varepsilon_{k+|\alpha|}(y)\int_{|y'|\leqq 1} \partial_i\partial^{\alpha_1}K(y-y')(1-\kappa^\varepsilon_{k+|\alpha_1|})\partial^{\alpha'}\omega(y')\,\mathrm{d}y' \\ \lesssim \varepsilon^{\beta|\alpha|+|\alpha'|}\int_{c\varepsilon^\beta\leqq|z|\leqq 2} |\partial_i\partial^{\alpha_1}K(z)|\,\mathrm{d}z\ \|\partial^{\alpha'}\omega\|_{L^p} \\ \lesssim \varepsilon^\beta\,\varepsilon^{|\alpha'|}\|\partial^{\alpha'}\omega\|_{L^p} \tag{141}$$

Hence,

$$\|I_{m,2}\|_{L^p} \lesssim \varepsilon^\beta\,\varepsilon^{|\alpha'|}\|\partial^{\alpha'}\omega\|_{L^p}. \tag{142}$$

Together with global estimates (60) and (139), this leads to

$$N^\varepsilon(C_2^{\alpha,\alpha'}) \lesssim \varepsilon^\beta\,\mathrm{e}^{\gamma t}. \tag{143}$$

We are now able to establish the final estimate. If we gather (129), (122), (130), (132), (133), (134) and (143) and the estimate (104) on the fundamental solution, we get

$$N^\varepsilon_{k,\mathrm{loc},m}(\omega) + \varepsilon N^\varepsilon_{k,\mathrm{loc},m}(\nabla\omega) \leqq C\mathrm{e}^{\gamma t}.$$

This ends the proof of (67).

It remains to explain bounds (68), (69) on the velocity. As in the proof of Lemma 4, we distinguish between the horizontal part v and v_3. The L^2 estimate on v is a direct consequence of (138), (142) with $p=2$ [together with (60), (67)]. For $p=\frac{2\mu-2}{\mu-2}$, these inequalities lead to

$$\|I_m\|_{L^{\frac{2\mu-2}{\mu-2}}} \lesssim \varepsilon^\beta\mathrm{e}^{\gamma t}. \tag{144}$$

Up to minor changes, one has in the same way

$$\|\nabla I_m\|_{L^{\frac{2\mu-2}{\mu-2}}} \lesssim \varepsilon^{\beta-1}\mathrm{e}^{\gamma t}. \tag{145}$$

To recover the L^∞ estimate on v, we use the Gagliardo–Nirenberg inequality

$$\begin{aligned}\|I_m\|_{L^\infty} &\leqq \|I_m\|^{1-a}_{L^{\frac{2\mu-2}{\mu-2}}} \|\nabla I_m\|^a_{L^{\frac{2\mu-2}{\mu-2}}}, \quad a = \frac{\mu-2}{\mu-1}, \\ &\leqq C\,\varepsilon^{\beta-a}\,\mathrm{e}^{\gamma t} \end{aligned} \tag{146}$$

thanks to inequalities (144), (145), and $\beta \geqq a$.

The inequality (69) on v_3, as in the proof of Lemma 4, is obtained in a similar (and even simpler) way as (67). The L^∞ bound follows again from (110). This concludes the proof of Lemma 5.

3. Centrifugal instability

In this section, we shall prove Theorem 2. The general scheme is the same as for Theorem 1. For the sake of brevity, we will only stress the main differences between the two proofs.

3.1. Notations

Because of our assumptions on the geometry of the streamlines, we shall recall the expression of the operators in curved coordinates. We consider the vicinity of the streamline ρ_0, and use the coordinates

$$(\rho,\theta) \to y(\rho,\theta) = (x_1(\theta,\rho), x_2(\theta,\rho)),$$

where ρ is chosen such that

$$\psi^s(y(\rho,\theta)) = \rho$$

and the map $\theta \to x_h(\rho,\theta)$ is $T(\rho)$ periodic. Precisely, we choose a parameterization $\theta \mapsto y_0(\theta)$ of the streamline $\rho = \rho_0$, and $y(\rho,\theta)$ satisfying the ODE

$$\partial_\rho y(\rho,\theta) = \frac{\nabla\psi_s(y(\rho,\theta))}{|\nabla\psi_s(y(\rho,\theta))|^2}, \quad y(\rho_0,\theta) = y_0(\theta).$$

A local orthonormal basis (e_ρ, e_θ) is then given by

$$e_\theta = \frac{1}{h(\rho,\theta)}\partial_\theta y, \quad e_\rho = |u^s|\partial_\rho y.$$

In this new basis, we can write

$$u^s = U^s(\rho,\theta)e_\theta$$

It will be also convenient to replace (ρ,θ) by (r,s) the curvilinear coordinates such that

$$\partial_s = \frac{1}{h}\partial_\theta, \quad \partial_r = U^s\partial_\rho.$$

To express the differential operator in this coordinates system, it is also useful to introduce the curvatures $(\gamma_\rho, \gamma_\theta)$ such that

$$\partial_r e_\rho = \gamma_\rho e_\theta, \quad \partial_r e_\theta = -\gamma_\rho e_\rho, \quad \partial_s e_\rho = \gamma_\theta e_\theta, \quad \partial_s e_\theta = -\gamma_\theta e_\rho.$$

By computing $\partial^2_{\rho\theta} y$ and $\partial_{\theta,\rho} y$, we easily get that

$$\gamma_\rho = \frac{\partial_\theta U^s}{hU^s}, \quad \gamma_\theta = U^s \frac{\partial_\rho h}{h}.$$

In the basis (e_ρ, e_θ, e_z), we have the expressions of the operators:

– the gradient

$$\nabla f = \begin{pmatrix} \partial_r f \\ \partial_s f \\ \partial_z f \end{pmatrix},$$

– the divergence of a vector field $v = (v_\rho, v_\theta, v_z) = v_\rho e_\rho + v_\theta e_\theta + v_z e_z$,

$$\mathrm{div} v = \partial_r v_\rho + \partial_s v_\theta + \partial_z v_z + \gamma_\theta v_\rho - \gamma_\rho v_\theta,$$

– the differential of a vector field:

$$Dv = \begin{pmatrix} \partial_r v_\rho - \gamma_\rho v_\theta & \partial_s v_\rho - \gamma_\theta v_\theta & \partial_z v_\rho \\ \partial_r v_\theta + \gamma_\rho v_\rho & \partial_s v_\theta + \gamma_\theta v_\rho & \partial_z v_\theta \\ \partial_r v_z & \partial_s v_z & \partial_z v_z \end{pmatrix}.$$

– the scalar Laplacian:

$$\Delta f = \partial_r^2 f + \partial_s^2 f + \partial_z^2 f + \gamma_\theta \partial_r f - \gamma_\rho \partial_s f$$

– the Laplacian of a vector field

$$\Delta v = (I \otimes \Delta) v + 2 D_1 v + D_0 v,$$

where Δ is the scalar Laplacian already defined and

$$D_1(\partial) = \begin{pmatrix} 0 & -(\gamma_\rho \partial_r + \gamma_\theta \partial_s) & 0 \\ \gamma_\rho \partial_r + \gamma_\theta \partial_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} \partial_r \gamma_\theta - \partial_s \gamma_\rho & -(\partial_r \gamma_\rho + \partial_s \gamma_\theta) & 0 \\ \partial_r \gamma_\rho + \partial_s \gamma_\theta & \partial_r \gamma_\theta - \partial_s \gamma_\rho & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we also notice that

$$u^s \cdot \nabla v + v \cdot \nabla u^s = U^s \partial_s v + M_0 v,$$

where

$$M_0 = \begin{pmatrix} -\gamma_\rho U^s & -2\gamma_\theta U^s & 0 \\ W & \partial_s U^s & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where W is defined as $\mathrm{curl} u^s = W e_z = (\partial_r U^s + \gamma_\theta U^s)\, e_z$.

3.2. WKB expansion

We keep the same notations as in Paragraph 2.1. Again, we look for approximations of the form (20). The main change is the choice of profiles U^k, which now involve the curved coordinates ρ, θ. They read

$$U^k(t,x) = \mathcal{U}^k\left(t, \frac{\rho-\rho_0}{\sqrt{\varepsilon}}, \theta, \frac{z}{\varepsilon}\right) + u^k(t,y)$$
$$= \sum_{l=0}^{2(N+1)} \sqrt{\varepsilon}^l\, \mathcal{U}^{k,l}\left(t, \frac{\rho-\rho_0}{\sqrt{\varepsilon}}, \theta, \frac{z}{\varepsilon}\right) + u^k(t,y),$$

where ρ_0 is given by (H1$'$), and $\mathcal{U}^{k,l} = (\mathcal{V}^{k,l}, \varepsilon\mathcal{P}^{k,l}) = \mathcal{U}^{k,l}(t, R, \theta, \lambda)$ is compactly supported in $[-1, 1]$ in R, periodic in λ, satisfying

$$\int \mathcal{U}^{k,l}(t, R, \theta, \lambda)\, \mathrm{d}\lambda = 0.$$

Finally, u^k is an average term issued from quadratic interactions. We give a few elements on the derivation of the $\mathcal{U}^k$.

Construction of $\mathcal{U}^0$. We Taylor expand the base flow u^s as

$$u^s(x) = u^s(x(\rho_0,\theta)) + O\left(\sqrt{\varepsilon}\right) = \left(U^s e_\theta\right)(\rho_0,\theta) + O\left(\sqrt{\varepsilon}\right).$$

We obtain

$$u^0 = 0, \quad \mathcal{V}_z^{0,0} = 0, \quad \mathcal{P}^{0.0} = 0,$$

and by using the expressions of the operator in the curved coordinates system

$$\partial_t \mathcal{V}_h^{0.0} + U^s(\rho_0,\theta)\partial_s \mathcal{V}_h^{0.0} + M_0(\rho_0,\theta)\mathcal{V}_h^{0.0} - \nu\partial_\lambda^2 \mathcal{V}_h^{0.0} = 0. \tag{147}$$

The equations on $\mathcal{U}^{0.k}$, $k \geqq 1$, are similar. The result that substitutes Lemma 3 is:

Lemma 7. *Thanks to* (H1$'$)*, there exists* $\mathcal{U}^0$ *which solves* (21) *smooth and such that we have*

$$\mathrm{e}^{\sigma t} \leqq \|\mathcal{V}^0(t)\| \leqq C\mathrm{e}^{\sigma t}, \quad t \in [0, T_\varepsilon].$$

Moreover, there exists ε_0 *such that for every* $\varepsilon \leqq \varepsilon_0$*, and for every* α *we have the estimate*:

$$|\partial_{R,\theta}^\alpha \mathcal{U}^0(t,R,\theta,\lambda)| \leqq C_{\alpha,N}\mathrm{e}^{\sigma t} \quad \forall Y \in D(0,1),$$
$$\left|\partial_{R,\theta}^\alpha \left(\mathcal{V}_z^0(t,R,\theta,\lambda)\right)\right| \leqq C_{\alpha,N}\sqrt{\varepsilon}\,\mathrm{e}^{\sigma t} \quad \forall Y \in D(0,1),$$

and

$$|\partial_Y^\alpha R^0(t,Y,\lambda)| + |\partial_Y^\alpha d^0(t,Y,\lambda)| \leqq C_{\alpha,N}\,\mathrm{e}^{\sigma t}, \quad t \in [0, T_\varepsilon].$$

Sketch of proof: As for the elliptic instability, the first step is to choose a growing solution of (147). We look for

$$\mathcal{V}^{0,0}(t, R, \theta, \lambda) = a(t, R, \theta)\, \mathrm{e}^{i\lambda} + c.c., \quad a_3^0 = 0.$$

Integration along the characteristics yields

$$\begin{aligned}\partial_t \left(a(t, R, \theta(t))\right) &= -M_0\, a(t, R, \theta(t)) - \nu a(t, R, \theta(t)) \\ &= \left(G(t, \theta_0, \rho_0) - \nu\right) a(t, R, \theta(t)),\end{aligned}$$

where G is defined in (13). By the same reasoning as before, thanks to (H1′), we end up with,

$$a(t, R, \theta) = \mathrm{e}^{\sigma t} P(t, \theta) \chi(R),$$

with a smooth P, that is periodic in t, and uniformly bounded in (t, θ), as well as its derivatives. It is clear that $a(t, R, \theta) \cdot e_\rho$ cannot vanish identically on any open interval, otherwise we would have, from the expression of M_0 and above equation, $a(t, R, \theta) \equiv 0$ on this open interval. This explains the amplification of the transverse component of v^ε in Theorem 2. Once this choice is made, the construction and control of the next profiles $\mathcal{V}^{0,k}$ follows as in Paragraph 2.2.

Construction of $\mathcal{U}^k$, $k \geqq 1$. The method is very close to the one used for elliptic instability and the reader is reffered to Paragraph 2.3. The main point is the treatment of the nonoscillating terms u^k, which are generated by nonlinear interactions. They solve again bidimensional Navier–Stokes equations, with right-hand side including terms of the type $F\left(t, \frac{\rho-\rho_0}{\sqrt{\varepsilon}}, \theta\right)$ (see expressions $Q^{k,1}$, $Q^{k,2}$ and $Q^{k,3}$). To control these mean profiles u^k, we must study again equations (54)and (55), and introduce adapted weighted norms. In the centrifugal case, they read

$$\begin{aligned}M^\varepsilon(f) &= \varepsilon^{-1/4}\left(\|f\|_{L^1} + \|f\|\right) + \|f\|_{L^4}, \\ X^\varepsilon(f) &= \varepsilon^{-1/4}\|f\| + \|f\|_{L^\infty}.\end{aligned}$$

As in Paragraph 2.4, we must introduce norms that control higher order derivatives. The global norms are

$$\begin{aligned}M^\varepsilon_{\mathrm{glob},m}(f) &= \sum_{|\alpha| \leqq m} \varepsilon^{|\alpha|}\, M^\varepsilon(D^\alpha f), \\ X^\varepsilon_{\mathrm{glob},m}(f) &= \sum_{|\alpha| \leqq m} \varepsilon^{|\alpha|}\, X^\varepsilon(D^\alpha f).\end{aligned}$$

To define local norms, we must account for the anisotropy between the ρ and θ variables. Due to the shape of the source terms, we should control $\sqrt{\varepsilon}^{|\alpha|}\, \partial_\theta^\kappa D^\alpha$ derivatives, and not only $\sqrt{\varepsilon}^{|\alpha|+\kappa}\, \partial_\theta^\kappa D^\alpha$ derivatives as in the elliptic case. On the other hand, we will not need to truncate in a vicinity $O(\sqrt{\varepsilon})$ of the streamline $\rho = \rho_0$ since in the centrifugal instability, we use the exact geometry of the streamlines, which is known in a vicinity of ρ_0 independent of ε, that is, for $\rho - \rho_0 = O(1)$. We therefore set:

$$M^{\varepsilon}_{k,\mathrm{loc},m}(f) = \sum_{2(|\alpha|+\kappa)+|\alpha'|\leq m} \sqrt{\varepsilon}^{|\alpha|}\,\varepsilon^{|\alpha'|}\,\chi_{k+|\alpha|+\kappa}\, M^{\varepsilon}\left(\partial^{\kappa}_{\theta} D^{|\alpha|+|\alpha'|} f\right),$$

$$X^{\varepsilon}_{k,\mathrm{loc},m}(f) = \sum_{2(|\alpha|+\kappa)+|\alpha'|\leq m} \sqrt{\varepsilon}^{|\alpha|}\,\varepsilon^{|\alpha'|}\,\chi_{k+|\alpha|+\kappa}\, X^{\varepsilon}\left(\partial^{\kappa}_{\theta} D^{|\alpha|+|\alpha'|} f\right),$$

where $\chi_k = \chi_k(\rho)$, $1 \leqq k \leqq K$ are truncation functions centered at ρ_0, satisfying (63). We obtain the following lemmas.

Lemma 8. (Global estimates) *Assume that the source term $F \in X$ enjoys for some $\gamma \geqq 2\sigma$ the estimate*

$$\|F\| + \|H\| \leqq C\varepsilon^{1/4}\mathrm{e}^{\gamma t} \tag{148}$$

with $H = \mathrm{curl} F = \partial_1 F_2 - \partial_2 F_1$ together with the global estimates

$$M^{\varepsilon}_{\mathrm{glob},m}(H) + \varepsilon^{-1/2} M^{\varepsilon}_{\mathrm{glob},m}(F_3) \leqq C_m \mathrm{e}^{\gamma t}. \tag{149}$$

Then, under the assumption (H2)*, we have for the vorticity $\omega = \mathrm{curl}\, v$, the global estimates*

$$M^{\varepsilon}_{\mathrm{glob},m}(\omega) + \varepsilon M^{\varepsilon}_{\mathrm{glob},m}(\nabla\omega) \leqq C_m \mathrm{e}^{\gamma t}. \tag{150}$$

Moreover, the velocity field **v** *enjoys the estimates*

$$X^{\varepsilon}_{\mathrm{glob},m}(\mathbf{v}) \leqq C_m \mathrm{e}^{\gamma t} \tag{151}$$

$$\varepsilon^{-1/2} M^{\varepsilon}_{\mathrm{glob},m}(v_3) + \varepsilon\varepsilon^{-1/2} M^{\varepsilon}_{\mathrm{glob},m}(\nabla v_3) \leqq C_m \mathrm{e}^{\gamma t}. \tag{152}$$

Lemma 9. (Local estimates) *Under the same assumptions as in the previous lemma, if moreover, the source term enjoys the local estimates*

$$M^{\varepsilon}_{k,\mathrm{loc},m}(H) + \varepsilon^{-1/2} M^{\varepsilon}_{k,\mathrm{loc},m}(F_3)\ \leqq C_m \mathrm{e}^{\gamma t}, \tag{153}$$

for k, m such that $k + m \leqq K$, then we also have for ω the local estimates

$$M^{\varepsilon}_{k,\mathrm{loc},m}(\omega) + \varepsilon M^{\varepsilon}_{k,\mathrm{loc},m}(\nabla\omega) \leqq C_m \mathrm{e}^{\gamma t}. \tag{154}$$

Moreover, the velocity field **v** *also enjoys the local estimates*

$$X^{\varepsilon}_{k,\mathrm{loc},m}(\mathbf{v}) \leqq C_m \mathrm{e}^{\gamma t}, \tag{155}$$

$$\varepsilon^{-1/2} N^{\varepsilon}_{k,\mathrm{loc},m}(v_3) + \varepsilon\varepsilon^{-1/2} M^{\varepsilon}_{k,\mathrm{loc},m}(\nabla v_3) \leqq C_m \mathrm{e}^{\gamma t}. \tag{156}$$

These lemma will be shown in the next paragraph. Proceeding exactly as in the elliptic case, we then obtain bounds similar to those of Lemma 6. The end of the proof of Theorem 2 follows as in Paragraph 2.6.

3.3. Proof of the lemmas

Global estimates To establish estimates (150) on ω, (151) on the horizontal part v of the velocity, and (152) on v_3 is a straightforward adaptation of what was done in the proof of Lemma 4. The L^∞ estimate on v_3 is recovered from (152) together with the Gagliardo–Nirenberg inequality

$$\|f\|_{L^\infty} \leqq C\,\|f\|_{L^4}^{1/2}\,\|\nabla f\|_{L^4}^{1/2}. \tag{157}$$

Local estimates The starting point of the proof is again equation (102) on the vorticity. We shall reason inductively on m. Note that the cases $m = 0, 1$ are covered by the global estimates. Assume now that the result holds for $m-1$. We introduce α, κ and α' such that $2(|\alpha|+\kappa)+|\alpha'| \leqq m$. We assume that $|\alpha|+\kappa \neq 0$, otherwise we use the global estimate (150). We shall estimate

$$W := \chi_{k+|\alpha|+\kappa}\left(U^s\partial_s\right)^\kappa D^{\alpha+\alpha'}\omega.$$

It is clearly equivalent to prove (154) or the estimate

$$\sqrt{\varepsilon}^{\alpha}\,\varepsilon^{|\alpha'|}\left(M^\varepsilon(W)+\varepsilon M^\varepsilon(\nabla W)\right) \lesssim \mathrm{e}^{\gamma t}. \tag{158}$$

The advantage of this choice is that $\chi_{k+\alpha+\kappa}\;(U^s\partial_s)$ commutes to the advection operator in (102). More precisely, W satisfies

$$\begin{aligned}
&\partial_t W + U^s\,\partial_s W - \nu\varepsilon^2\Delta W \\
&\quad = \chi_{k+|\alpha|+\kappa}\left(U^s\partial_s\right)^\kappa D^{\alpha+\alpha'} H - \left[D^{\alpha+\alpha'},\, U_s\partial_s\right]\chi_{k+|\alpha|+\kappa}\left(U^s\partial_s\right)^\kappa\omega \\
&\qquad -\chi_{k+|\alpha|+\kappa}\left(U^s\partial_s\right)^\kappa D^{\alpha+\alpha'}(v\cdot\nabla\omega_s) - \nu\varepsilon^2\left[\chi_{k+|\alpha|+\kappa}\left(U^s\partial_s\right)^\kappa,\,\Delta\right]D^{\alpha+\alpha'}\omega \\
&\quad = \sum_{j=1}^{4} I_j.
\end{aligned} \tag{159}$$

By assumption (153), we have

$$\sqrt{\varepsilon}^{|\alpha|}\,\varepsilon^{|\alpha'|}\,M^\varepsilon(I_1) \lesssim \mathrm{e}^{\gamma t}.$$

The commutator with the diffusion can be written

$$\begin{aligned}
I_4 &= \nu\varepsilon^2\left[\chi_{|\alpha|+\kappa},\,\Delta\right]\left(U^s\partial_s\right)^\kappa D^{\alpha+\alpha'}\omega + \nu\varepsilon^2\,\chi_{|\alpha|+\kappa}\left[\left(U^s\partial_s\right)^\kappa,\,\Delta\right]D^{\alpha+\alpha'}\omega \\
&= I_{4,1} + I_{4,2}.
\end{aligned}$$

The first term at the right-hand side satisfies, with (63),

$$\left|I_{4,1}\right| \leqq \chi_{k+|\alpha|+\kappa-1}\sum_{|\delta|\leqq 1}\left|D^\delta\left(U^s\partial_s\right)^\kappa D^{\alpha+\alpha'}\omega\right|$$

and thus, by the induction assumption,

$$\sqrt{\varepsilon}^{|\alpha|}\,\varepsilon^{|\alpha'|}\,M^\varepsilon(I_{4,1}) \lesssim \varepsilon\,M^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega) \lesssim \mathrm{e}^{\gamma t}.$$

Using the expression of the scalar Laplacian in two dimensions, we get

$$\left[\left(U^s \partial_s\right)^\kappa \ , \ \Delta\right] = \sum_{j=0}^{\alpha+1} a_j \, \partial^j \theta \ + \ \sum_{j=0}^{\alpha} b_j \, \partial_\theta^j \partial_\rho \ + \ \sum_{j=0}^{\alpha-1} c_j \, \partial_\theta^j \partial_\rho^2.$$

By playing again with the different derivatives, this leads to

$$\sqrt{\varepsilon}^{\alpha} \, \varepsilon^{|\alpha'|} \, M^\varepsilon(I_{4,2}) \ \lesssim \ \varepsilon \, M^\varepsilon_{k,\mathrm{loc},m-1}(\nabla\omega) \ \lesssim \ \mathrm{e}^{\gamma t}.$$

As for the elliptic instability, the nonlocal term I_3 is delicate, as ∂_s does not commute with $K*$, that is, the convolution by the Biot–Savart kernel. Let us introduce the streamfunction ψ such that $v = \nabla^\perp \psi$. We recall that

$$\psi(y) = H * \omega(y) \ := \ \frac{1}{2\pi} \int \ln|y - y'| \omega(y') \, \mathrm{d}y', \quad \Delta\psi = \omega.$$

As in the proof of Lemma 5, we write

$$v(x) = v_r(x) \ + \ v_s(x), \ \ v_r(x) = K * \left((1 - \chi_{k+|\alpha|+\kappa-1})\omega\right) + K * \left(\chi_{k+|\alpha|+\kappa-1}\omega\right).$$

By property (63), we have easily for all p

$$\|\chi_{k+|\alpha|+\kappa} \partial_\theta^\kappa D^{\alpha+\alpha'} v_r\|_{L^p} \leqq C\|\omega\|_{L^p} \leqq C\mathrm{e}^{\gamma t},$$

as the singular part of K is removed. We can therefore assume without loss of generality that $\chi_{k+|\alpha|+\kappa-1}\omega = \omega$ and study the second term v_s. Note also that the derivative $D^{\alpha+\alpha'}$ commutes with $K*$, so that it is enough to treat the case $\alpha = \alpha' = 0$. It thus remains to evaluate $\chi_{k+\kappa} \partial_\theta^\kappa v$. We now prove that, for all $1 \leqq j \leqq \alpha$,

$$M^\varepsilon \left(\chi_{k+j} \, \nabla^\perp \, \partial_s^j \, \left(\chi_{k+j-1}\psi\right)\right) \lesssim \mathrm{e}^{\gamma t}. \tag{160}$$

Note that such a bound easily implies that

$$M^\varepsilon \left(\chi_{k+\kappa} \partial_s^\kappa v\right) = M^\varepsilon \left(\chi_{k+\kappa} \partial_s^\kappa \nabla^\perp \psi\right) \ \lesssim \ \mathrm{e}^{\gamma t}, \quad \sqrt{\varepsilon}^{|\alpha|} \, \varepsilon^{|\alpha'|} M^\varepsilon(I^3) \ \lesssim \ \mathrm{e}^{\gamma t}.$$

We have

$$\Delta\left(\chi_{k+j-1}\psi\right) = \chi_{k+j-1}\omega + 2\nabla\chi_{k+j-1} \cdot \nabla\psi + \Delta\chi_{k+j-1}\psi$$

so that

$$\begin{aligned} \Delta\left(\partial_s^j \, \chi_{k+j-1}\psi\right) \ &= \ \partial_s^j \left(\chi_{k+j-1}\omega\right) + \partial_s^j \left(2\nabla\chi_{k+j-1} \cdot \nabla\psi + \Delta\chi_{k+j-1}\psi\right) \\ &\quad + [\Delta, \partial_s^j] \left(\chi_{k+j-1}\psi\right) \\ &:= F_j. \end{aligned}$$

Thus,

$$\chi_{k+j} \, \nabla^\perp \, \partial_s^j \, \left(\chi_{k+j-1}\psi\right) = \chi_{k+j} \, K * F_j.$$

Let us treat the case $j = 1$. The source term F_1 involves derivatives of ω up to the first order, derivatives of ψ up to the second order, and is zero outside the support of χ_k. One may write

$$K * F_1 = a\, K * \omega + b\, \nabla(K * \omega) + a'\, K * \tilde{\psi} + b'\, \nabla(K * \tilde{\psi}) + c'\, \nabla\left(K * \nabla\tilde{\psi}\right),$$

where $\tilde{\psi} := \chi_{k-1}\psi$, whereas a, b and a', b', c' denote smooth (matrix) coefficients. We deduce easily from the properties of K and H that

$$\begin{aligned} &\|\chi_{k+1}\, K * F_1\|_{L^1} \;+\; \|\chi_{k+1}\, K * F_1\| \\ &\quad\lesssim\; \|\omega\|_{L^1} + \|\omega\| + \|\tilde{\psi}\|_{L^1} + \|\tilde{\psi}\| + \|\nabla\tilde{\psi}\|_{L^1} + \|\nabla\tilde{\psi}\| \\ &\quad\lesssim\; \|\omega\|_{L^2} \;\lesssim\; \varepsilon^{1/4}\, \mathrm{e}^{\gamma t} \end{aligned}$$

using for the last inequality estimate (150). Similarly,

$$\|\chi_{k+1}\, K * F_1\|_{L^4} \;\lesssim\; \|\omega\|_{L^4} \;\lesssim\; \mathrm{e}^{\gamma t},$$

which shows (160) for $j = 1$. The inequality for general $1 \leqq j \leqq \kappa$ follows inductively in the same way.

It remains to handle I_2. Let us first consider the case $|\alpha| = 0$. Either $|\alpha'| = 0$ and $I_2 = 0$, or $\alpha'| \geqq 1$, $2\kappa + |\alpha'| \leqq m$,

$$I_2 = \sum_{\tilde{\alpha}} C_{\tilde{\alpha}}\, D^{\tilde{\alpha}}\, \left(\chi_{k+\kappa}\left(U^s\partial_s\right)^{\kappa} \omega\right),$$

where $|\tilde{\alpha}| \leqq |\alpha'|$, and so

$$\varepsilon^{|\alpha'|}\, M^{\varepsilon}\,(I_2) \;\lesssim\; \varepsilon M^{\varepsilon}_{k,\mathrm{loc},m-1}(\nabla\omega) \;\lesssim\; \mathrm{e}^{\gamma t}.$$

From the bounds on I_1, I_3, I_4 (valid for all α, α') and by properties (104), we get that

$$\sum_{2\kappa+|\alpha'|\leqq m} \varepsilon^{|\alpha'|}\left(M^{\varepsilon}\left(\chi_{k+\kappa}\,\partial_\theta^{\kappa} D^{\alpha'}\omega\right) + \varepsilon M^{\varepsilon}\left(\nabla\left(\chi_{k+\kappa}\,\partial_\theta^{\kappa} D^{\alpha'}\omega\right)\right)\right) \lesssim \mathrm{e}^{\gamma t}. \quad (161)$$

Once (161) has been established, one can treat the case $|\alpha| = 1$. In this case, the annoying term reads

$$C_{\alpha,\alpha'}\sqrt{\varepsilon}\varepsilon^{|\alpha'|} D^{\tilde{\alpha}}\left(\chi_{k+1+\kappa}\left(U^s\partial_s\right)^{\kappa}\omega\right)$$

with $|\tilde{\alpha}| = |\alpha'| + 1$. We use

$$\begin{aligned} &\sqrt{\varepsilon}\varepsilon^{|\alpha'|} M^{\varepsilon}\left(D^{\tilde{\alpha}}\left(\chi_{k+1+\kappa}\left(U^s\partial_s\right)^{\kappa}\omega\right)\right) \\ &\quad\lesssim\; \sqrt{\varepsilon} M^{\varepsilon}_{k,\mathrm{loc},m-1}(\omega) + \sqrt{\varepsilon}\varepsilon^{|\alpha'|} \sum_{2(\kappa+1)+|\alpha'|\leqq m} M^{\varepsilon}(\chi_{k+1+\kappa} D^{\alpha'}\partial_\theta^{\kappa+1}\omega) \\ &\quad\lesssim\; \sqrt{\varepsilon}\, \mathrm{e}^{\gamma t}, \end{aligned}$$

using (161) to control the second term on the right-hand side With the same reasoning, one can go from $|\alpha| = j$ to $|\alpha| = j + 1$, so that

$$\sqrt{\varepsilon}^{\alpha}\, \varepsilon^{\alpha'}\, M^{\varepsilon}(I_2) \;\lesssim\; \mathrm{e}^{\gamma t}$$

in the general case. Using again the bounds on I_j and estimate (104), we deduce (158) and thus (154).

To complete the proof of Lemma 5, we still need the estimates on the velocity. The only difficulty is to control in L^∞ the horizontal part v of the velocity. In other words, we must control $\|\chi_{k+\kappa}\partial_s^\kappa v\|_{L^\infty}$. We recall that inequality (99) holds replacing $v = K * \omega$ by the integral operator by $\tilde{v} = \tilde{K} * \omega$ if $\tilde{K}$ is bounded by an homogeneous function of degree -1. For any smooth vector field $a = a(y)$ with compact support, we now have

$$\begin{aligned} a(y)\cdot\nabla v(y) &= a(y)\cdot\nabla\int K(z)\omega(y-z)\,\mathrm{d}z \\ &= \int K(z)a(y)\cdot\nabla\omega(y-z)\,\mathrm{d}z \\ &= \int K(z)a(y-z)\cdot\nabla\omega(y-z)\,\mathrm{d}z \\ &\quad + \int K(z)(a(y)-a(y-z))\nabla\omega(y-z)\,\mathrm{d}z. \end{aligned}$$

The first term is simply $K * (a\cdot\nabla\omega)(y)$. After integration by parts, the second term is bounded by

$$\left|\int K(z)(a(y)-a(y-z))\nabla\omega(y-z)\,\mathrm{d}z\right| \;\leqq\; M\,\tilde{K} * |\omega|(z)|,$$

where $M = M(\|\nabla a\|_{L^\infty})$, and $\tilde{K}(z) := |K(z)| + |z|\,|\nabla K(z)|$. Thanks to above remarks, we get

$$\|a(y)\cdot\nabla v\|_{L^\infty} \;\leqq\; C\left(\|a(y)\cdot\nabla\omega\|_{L^p} + \|a(y)\cdot\nabla\omega\|_{L^q} \;+\; \|\omega\|_{L^p} \;+\; \|\omega\|_{L^q}\right).$$

for $p < 2$, $q > 2$. Specializing $p = 1$, $q = 4$ and $a\cdot\nabla = \chi_{k+1}\partial_s$ we deduce from (154)

$$\|\chi_{k+1}\partial_s v\|_{L^\infty} \;\lesssim\; \mathrm{e}^{\gamma t}.$$

More generally, one can show recursively [applying $(a\cdot\nabla)^j = \left(\chi_{k+j}\partial_\theta\right)^j$] that:

$$\|\chi_{k+j}\partial_s^j v\|_{L^\infty} \;\lesssim\; \mathrm{e}^{\gamma t}, \quad 1 \leqq j \leqq \kappa.$$

This concludes the proof.

4. Examples

This section is devoted to various applications of Theorems 1 and 2. In order to include more examples of centrifugal instabilities, we can consider the Navier–Stokes equations with Coriolis forcing

$$\partial_t u + u \cdot \nabla u + \nabla p + \Omega e_z \times u = \nu \varepsilon^2 \Delta u + F, \quad \nabla \cdot u = 0 \tag{162}$$

where Ω is the rotation speed. All results established on (1) extend without any difficulty to (162). Indeed, the bicharacteristic-amplitude system (7), (8), (9) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u^s(x), \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = -\nabla u^s \cdot \xi, \\ \dfrac{\mathrm{d}a}{\mathrm{d}t} = \left(2\dfrac{\xi \otimes \xi}{|\xi|^2} - I \right) Du^s \cdot a + \Omega \left(\dfrac{\xi \otimes \xi}{|\xi|^2} - I \right) e_z \times a \end{cases}, \tag{163}$$

so that it suffices to check (H1) or (H1′) on this system. Note that the most technical part in the proof of the nonlinear instability theorem, which was the analysis of the linearized two-dimensional Navier–Stokes equation (54) summarized in Lemmas 4, 5 and 8, 9, does not change since in two dimensions, the Coriolis force term $e_z \times v$ can be written as a gradient and hence incorporated into the pressure.

4.1. Elliptic instability

The aim of this section is to give an example of a two-dimensional flow u^s with a stagnation point for which both the assumptions (H1) and (H2) are verified so that Theorem 1 applies. We still consider the case without Coriolis force $\Omega = 0$.

As a first step, we shall prove that when the streamlines in the vicinity of a stagnation point are really elliptic the assumption (H1) can be checked analytically by reducing the problem to an Ince equation as was shown by WALEFFE [27].

Let us assume that A is under the form

$$Ax = \begin{pmatrix} 0 & -(1-\delta) \\ 1+\delta & 0 \end{pmatrix}, \quad \delta \in (0, 1). \tag{164}$$

Note that when $\delta = 0$, the streamlines are circles. We have the following result:

Proposition 10. *For every* δ, $\delta \in (0, 1)$, *there exists* ξ_0 *such that* $\sigma_0 > 0$, *that is,* (H1) *is verified.*

Proof. We can decompose A under the form

$$A = R + \delta J, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A phase ξ satisfying equation (8) is

$$\xi_h = \mu\sqrt{\frac{1+E^2}{2}}\begin{pmatrix}\cos\omega t\\ E\sin\omega t\end{pmatrix}, \quad \xi_3 = \lambda\sqrt{\frac{1+E^2}{2}},$$

where ω and the eccentricity E are defined by

$$\omega = \sqrt{1-\delta^2}, \quad E = \sqrt{\frac{1-\delta}{1+\delta}},$$

and λ and μ are free parameters which will be chosen later.

To prove the proposition, we shall reduce the equation for the amplitude to an equivalent second-order ordinary differential equation, the Ince equation, which is well studied in the literature. The computations are easier if we keep the physical quantities so that we first rewrite (9) by using the pressure, an equivalent formulation is:

$$a' = -Aa - i\xi p, \tag{165}$$

$$a \cdot \xi = 0 \tag{166}$$

so that the pressure is given by

$$-i|\xi|^2 p = -2Aa\cdot\xi = -2a_h \cdot R\xi_h + 2\delta a_h \cdot J\xi_h. \tag{167}$$

Next, we easily get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_h \cdot R\xi_h) = -2a_h \cdot \xi_h \tag{168}$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_h \cdot J\xi_h) = -2\,i\,p\,\xi_1\xi_2 - 2\,\delta\,a_h\cdot\xi_h. \tag{169}$$

To conclude, we first use (166) and (165),

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_h\cdot\xi_h) = -\xi_3\frac{\mathrm{d}a_3}{\mathrm{d}t} = i\xi_3^2 p$$

and hence, by using (167) and (168), (169), we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-|\xi|^2\frac{\mathrm{d}}{\mathrm{d}t}(a_h\cdot\xi_h)\right) = -4\delta\xi_1\xi_2\frac{\mathrm{d}}{\mathrm{d}t}(a_h\cdot\xi_h) + 4\omega^2\xi_3^2\, a_h\cdot\xi_h.$$

To get an equation in a convenient form, we set

$$\alpha = \frac{\delta\mu^2}{\lambda^2+\mu^2}, \quad c = \frac{4\lambda^2}{\lambda^2+\mu^2}$$

and $a_h\cdot\xi_h(t) = y(\omega t)$ with $\tau = \omega t$ so that we find the Ince equation

$$(1+\alpha\cos 2\tau)y'' - 4\alpha\sin(2\tau)y' + cy = 0, \tag{170}$$

where $'$ stands for the derivative with respect to τ with parameters $c \in (0, 4)$ and $\alpha(\delta) \in (0, 1)$ since λ and μ were free. Note that c plays the part of a spectral parameter and that c does not depend on δ. This equation can be transformed into a classical Hill equation; by setting

$$y = \frac{z}{1 + \alpha \cos 2\tau},$$

we get

$$(1 + \alpha \cos 2\tau)z'' + (4\alpha \cos 2\tau + c)z = 0. \tag{171}$$

To prove the proposition, it suffices to prove that, for all values of α in $(0, 1)$, we can find some unstable eigenvalue c for the spectral problem (170). The classical theory of the Hill equation (we refer to [18]) gives the existence of two sequences c_i and c_i' such that there exists a periodic solution of period π for $c = c_i'$ and a periodic solution of period 2π for $c = c_i$. Moreover, we have

$$c_0' < c_1 \leqq c_2 < c_1' \leqq c_2' < c_3 \leqq c_4 \cdots$$

and the equation has unstable solutions in the intervals (c_{2i-1}, c_{2i}) and (c_{2i-1}', c_{2i}'). Next, since for $c = 0$, the constant solution 1 is obviously a solution of (170) which does not vanish and hence, thanks to [18], Theorem 4.1, we get that $c_0' = 0$. Moreover, we also see on the form (171) that for $c = 4$, we have π periodic solutions and hence $c_1' \leqq 4$ so that the interval of instability (c_1, c_2) is in the range of the parameter c. To conclude, it suffices to check that this interval of instability is not empty. Let us define the polynomials

$$Q(\mu) = 2\alpha\mu^2 + 4\alpha\mu, \quad Q^*(\mu) = 2Q\left(\mu - \frac{1}{2}\right).$$

For the Ince equation, we have the following characterization of coexistence, we refer to [18] Theorem 7.1:

Lemma 11. *If* (170) *has two linearly independent solutions of period* 2π, *then* $Q^*(\mu)$ *vanishes for some* $\mu \in \mathbb{Z}$.

By using this Lemma, we get that the intervals of instability (c_{2i-1}, c_{2i}) are nonempty for every $\alpha \in (0, 1)$. Consequently, we can always find an unstable solution by choosing c in (c_1, c_2). This ends the proof of the proposition. □

The second step is to find a flow which verifies (H2). In the domain $\mathbb{T}^3$, we consider the flow given by the streamfunction

$$\varphi^s(x_1, x_2) = a \cos x_1 + b \cos x_2 \tag{172}$$

for $a,\ b \in \mathbb{R}$. These flows have both elliptic and hyperbolic stagnation points. We can prove:

Proposition 12. (H2) *is verified for the flows given by* (172) *for every* a, b *and* ε *sufficiently small in the sense that* (17) *holds if* (15) *is verified for every* $\gamma > 0$.

Proof. These flows are actually examples of the marginally stable flows studied in [12]. To study (16) in $\mathbb{T}^2$, we first estimate the mean $\overline{v} = \int_{\mathbb{T}^2} v$. Thanks to (16), we get

$$\partial_t \overline{v} = \int f$$

and hence

$$|\overline{v}(t)| \leqq C\mathrm{e}^{\gamma t}. \tag{173}$$

Now, we can set $v = \overline{v} + \tilde{v}$ with $\int_{\mathbb{T}^2} \tilde{v} = 0$ and $\tilde{v}$ will be a solution of

$$\partial_t \tilde{v} + u^s \cdot \nabla \tilde{v} + \tilde{v} \cdot \nabla u^s - \nu\varepsilon^2 \Delta \tilde{v} = g, \tag{174}$$

with

$$g = f - \overline{v} \cdot \nabla u^s.$$

We note that the assumption (15) is still verified on g. By using the vorticity $\omega = \partial_1 \tilde{v}_2 - \partial_2 \tilde{v}_1$, and the relation $\psi^s = -\omega^s$, we can rewrite (174) as

$$\partial_t \omega = \{\omega^s, L\omega\} + \nu\varepsilon^2 \Delta\omega + F \tag{175}$$

where $F = \mathrm{curl} g$, L is the linear operator defined by

$$L\omega = \Delta^{-1}\omega + \omega, \quad \Delta^{-1}\omega = -\sum_{(k,l)\neq(0,0)} \frac{\hat{\omega}_{k,l}}{k^2 + l^2} \mathrm{e}^{ikx_1 + ilx_2}$$

and $\{f, g\}$ is the Poisson bracket:

$$\{f, g\} = \partial_1 f \, \partial_2 g - \partial_2 f \, \partial_1 g.$$

Next, we notice that L is a bounded self-adjoint operator and that, thanks to the Fourier series, we have the orthogonal decomposition

$$L^2 = H \oplus E, \quad H = \mathrm{Ker} L = \langle \cos x, \cos y, \sin x, \sin y \rangle \tag{176}$$

where E is a stable subspace and $L_{/E}$ is positive:

$$\exists \alpha > 0 \ \forall \omega \in E, \quad \int L\omega\omega \geqq \alpha \|\omega\|^2; \tag{177}$$

Moreover, for this simple example, H and E are also stable by the Laplacian.

Thanks to (175), we immediately get by using the decomposition

$$\omega = \omega_0 + \omega_1, \quad \omega_0 \in H, \ \ \omega_1 \in E$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int L\omega_1\omega_1 + \nu\varepsilon^2 ||\nabla\omega_1||^2 = \nu\varepsilon^2 \|\omega_1\|^2 + \int F L\omega_1$$

and hence we get, thanks to (177),

$$\|\omega_1(t)\|^2 \leqq \frac{\nu\varepsilon^2}{\alpha}\int_0^t \|\omega_1\|^2 + C\int_0^t \|F(s)\|\,\|\omega_1(s)\|\,\mathrm{d}s.$$

To integrate this inequality, we set

$$z(t)^2 = \frac{\nu\varepsilon^2}{\alpha}\int_0^t \|\omega_1\|^2 + C\int_0^t \|F(s)\|\,\|\omega_1(s)\|\,\mathrm{d}s,$$

we find

$$z' \leqq \frac{\nu\varepsilon^2}{\alpha} z + C\|F\|,$$

and hence we get

$$z(t) \leqq C\int_0^t \mathrm{e}^{2\sigma s}\mathrm{e}^{\nu\varepsilon^2/\alpha(t-s)}\mathrm{d}s.$$

For ε sufficiently small such that $\nu\varepsilon^2/\alpha < 2\sigma$, this finally yields

$$\|\omega_1(t)\| \leqq \frac{C}{2\sigma - \nu\varepsilon^2/\alpha}\mathrm{e}^{\gamma t}. \tag{178}$$

It remains to estimate ω_0. Thanks to (175), we have

$$\partial_t\omega_0 = P\{\omega^s, L\omega_1\} + \nu\varepsilon^2\Delta\omega_0 + PF,$$

where P is the orthogonal projection on H. Thanks to an integration by parts, this yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_0\|^2 \leqq C\|\omega_1\|\,\|\nabla\omega_0\| + \|F\|\,\|\omega_0\|.$$

Since H is a finite-dimensional space generated by smooth functions, we have $\|\nabla\omega_0\| \leqq C\|\omega_0\|$. Consequently, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_0\| \leqq C\|\omega_1\| + \|F\|$$

and hence thanks to (178), we get

$$\|\omega_0\| \leqq C\mathrm{e}^{\gamma t}. \tag{179}$$

By collecting (173), (178), (179), we get (17). □

Finally, by combining Propositions 10 and 12, we easily get an example for which Theorem 1 holds. We consider the flow (172) with $a = -(1-\delta)$ and $b = -(1+\delta)$, then for $\delta \in (0, 1)$, $(0, 0)$ is an elliptic stagnation and $Du^s(0, 0)$ is under the form (164). Consequently we have:

Proposition 13. *For the Taylor–Green flow* (172) *with* $a = -(1-\delta)$ *and* $b = -(1+\delta)$, (H1) *and* (H2) *are verified for every* $\delta \in (0, 1)$ *and hence the result of Theorem* 1 *holds.*

Remark 14. Throughout the paper, we have considered the nonhomogeneous Navier–Stokes equation (2) with a well-chosen exterior forcing in order to study the stability of stationary solutions and one can wonder if this is a severe restriction. This problem can be easily discussed on the example of the Taylor–Green flow (172). Indeed, this is a stationary solution of the Euler equation but not of the Navier–Stokes equation. The vorticity of the solution of the homogeneous Navier–Stokes equation with initial value (172) is given explicitly by

$$\omega(t) = \mathrm{e}^{-\nu\varepsilon^2 t}(a\cos x_1 + b\cos x_2) \tag{180}$$

and hence, we see that this solution can be considered as stationary in the sense that it is very well approximated by (172) for times of order $t \ll \varepsilon^{-2}$. Since the nonlinear instability that is described by Theorem 1 requires to follow the behavior of the system up to times of order $|\log\varepsilon|$, it is very legitimate to neglect the slow motion of (180) in this stability study.

4.2. Centrifugal instability

4.2.1. Circular vortices The simplest example of centrifugal instability is the one of circular vortices: the velocity field u^s is given by

$$u^s(x) = U(r)e_\theta, \tag{181}$$

where we have used the standard polar coordinates ($y = (r\cos\theta, r\sin\theta)$) in the plane.

Proposition 15. *Assume that there exists r_0 such that*

$$(W(r) + \Omega)\left(\Omega + \frac{2U}{r}\right) < 0, \tag{182}$$

then assumptions (H1′) *is verified. Moreover,* (H2) *is also verified and hence Theorem* 2 *holds.*

Note that there is room for instability even in the absence of external rotation ($\Omega = 0$); in this case, if we assume that $U > 0$ the criterion is just the existence of a streamline with $W(r_0) < 0$, that is to say with negative vorticity. Moreover, we also note that there exists flows which are stable with respect to two-dimensional perturbations (that is, flows such that $W' < 0$) which can undergo centrifugal instability.

Proof. We consider a phase given by $\varphi(z) = z$ so that $\xi(t) = e_z$ does not depend on time. We also notice that the equation (7), reduces to

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = U(r), \quad \frac{\mathrm{d}r}{\mathrm{d}t} = 0$$

so that on an integral curve r does not depend on time also. With this choice of phase, the amplitude equation (9) reduces to a_3 =0 and the simple constant coefficient ordinary differential equation

$$\frac{\mathrm{d}a_h}{\mathrm{d}t} = -M(r)a_h, \quad M(r) = \begin{pmatrix} 0 & -\Omega - \frac{2U(r)}{r} \\ W(r) + \Omega & 0 \end{pmatrix}.$$

Consequently on this example, our instability criterion is equivalent to the existence of an unstable eigenvalue for M_0 and hence, we find (182).

The next step is to check (H2). Since the curvature of the streamline is constant, we can choose

$$X = \{v = v(t, r)\}.$$

Consequently, to solve (16), since $f \in X$, we can also choose $v = V(t, r)e_\theta$, which yields

$$\partial_r p = f_r + 2\frac{U}{r},$$

which will determine p with $\partial_\theta p = 0$ and hence, we get for v the standard heat equation

$$\partial_t v - \nu\varepsilon^2 \Delta v = f_\theta(t, r). \tag{183}$$

Consequently, the estimate (17) is verified. □

Remark 16. For the simple example (181), if the criterion (182) is verified, the conclusion of Theorem 2 holds even if $\nu = 0$, that is for the Euler equation. Indeed, in the proof of Theorem 2 the presence of the viscosity is used only in order to get the estimates for high-order derivatives in Lemmas 8, 9. As explained before these estimates are in general false for the two-dimensional linearized Euler equation even if (H2) is verified. Nevertheless, as we have just seen, in the special case of circular vortices when $\nu = 0$, we just have to study one-dimensional Euler equation, which is just the simplest ODE $\partial_t v = f$. Consequently, if the source term satisfies the assumptions of Lemma 9 then v obviously satisfies the same estimates.

4.2.2. Parallel shear flows A related example is the case of planar shear flows

$$u^s(x) = (U(x_2), 0, 0)^t$$

in the domain $\Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. Again, for $\varphi(z) = z$, the amplitude equation reduces to

$$\frac{\mathrm{d}a_h}{\mathrm{d}t} = -M(x_2)a_h, \quad M(y) = \begin{pmatrix} 0 & -(W + \Omega) \\ 2\Omega & 0 \end{pmatrix}, \quad W = -U'(x_2).$$

The criterion for instability becomes: there exists x_2 such that $\Omega(W(x_2) + \Omega) < 0$. Note that in this case the presence of external rotation is necessary for the instability. Again, we can check as in the previous example that the assumption (H2) is verified since we can choose the space X to be made of functions of x_2 only so that the two-dimensional linearized Navier–Stokes equation reduces to a one-dimensional heat equation. Moreover, as for the circular vortices, for this simple example, Theorem 2 holds even if $\nu = 0$.

4.2.3. More-general flows Now we consider a general flow with closed streamlines such that $\omega^s = F(\psi^s)$. In this general case, the Floquet exponents of system (9) can be evaluated numerically. There are many more concrete criteria available in the literature. For example the generalized Bayly's criterion [1]:

Proposition 17. *If there is a streamline such that* $(\gamma_\theta U + \Omega)(W + 2\Omega) < 0$ *everywhere, then* (H1$'$) *is verified.*

This is a sufficient criterion but is not always verified on concrete examples. Another one characterizes the bifurcation of a neutral streamline towards instability. We assume that there is a streamline ρ_0 such that $W + 2\Omega = 0$, then

Proposition 18. *If* $W'(\rho_0)\int_0^T \left(\frac{\gamma_\theta}{U^s}(\theta(t), \rho_0) + \frac{\Omega}{(U^s)^2}\right) \mathrm{d}t > 0$ (*respectively* < 0) *then there exists* $\eta > 0$ *such that for every* $\rho \in (\rho_0, \rho_0 + \eta)$ (*resp.* $\rho \in (\rho_0 - \eta, \rho_0)$) *the streamline* ρ *is unstable, that is,* (H1$'$) *is matched.*

This criterion was conjectured in [23] and proved in [3].

Finally, if in addition, we assume that the flow verifies one of the inviscid Arnold stability criteria [19] then our assumption (H2) will be verified. In particular the *Taylor–Green* flow (172) gives again an interesting example for which Theorem 2 applies. Let us consider the case where $a = b = -1$ (in this case there is no elliptic instability), then (H1$'$) can be checked numerically see [3] for example, and Proposition 12 shows that (H2) is verified.

Appendix: $W^{1,p}$ estimates for advection–diffusion operators

Let $u = u(y)$ a smooth vector field, divergence free. We consider the scalar advection–diffusion equation

$$\partial_t\theta + u \cdot \nabla\theta - \nu\Delta\theta, \quad \theta|_{t=0} = \theta_0. \tag{184}$$

and denote by $S(t) : \theta_0 \mapsto \theta(t)$ the corresponding semigroup. The aim of this appendix is to establish the second estimate of (104); the first one, which states that S is a contraction in L^p for every p, is very classical. Note that for notational convenience we denote in this section the viscosity by ν and not by $\nu\varepsilon^2$.

Lemma 19. *Let* $u \in W^{1,\infty}$ *be a divergence-free vector field, then there exists* $C > 0$ *such that for all* $1 \leqq p \leqq +\infty$ *and for all* $\nu > 0$, *the semigroup* $S(t)$ *satisfies:*

$$\forall t \geqq 0, \quad \|\nabla S(t)\|_{\mathcal{L}(L^p)} \leqq \frac{C}{\sqrt{\nu}} \max\left(1, \frac{1}{\sqrt{t}}\right), \quad C = C(\|\nabla u\|_{L^\infty}). \tag{185}$$

The main interest of (185) is that the estimate holds globally in time with a sharp dependence in the viscosity. The fact that such an estimate is valid on a small interval of time $[0, T_\nu]$ is very classical for parabolic equations.

Proof. We first consider the case $p = 2$, which can be handled easily through energy bounds. Indeed, the standard energy estimate for (184) yields

$$\frac{1}{2}\partial_t \|\theta(t)\|^2 + \nu \|\nabla\theta(t)\|^2 = 0 \tag{186}$$

and also

$$\frac{1}{2}\partial_t \left(t\,\|\theta(t)\|^2\right) + \nu\, t\, \|\nabla\theta(t)\|^2 = \frac{1}{2}\|\theta(t)\|^2 \leqq \frac{1}{2}\|\theta_0\|^2. \tag{187}$$

The last inequality comes from (186). Moreover, multiplying (184) by $-\Delta\theta$, we obtain

$$\frac{1}{2}\partial_t \|\nabla\theta(t)\|^2 + \nu\|\Delta\theta(t)\|^2 = \int u\cdot\nabla\theta\,\Delta\theta.$$

Several integrations by parts lead to

$$\begin{aligned}\int u\cdot\nabla\theta\,\Delta\theta &= -\int (\nabla u\cdot\nabla\theta)\cdot\nabla\theta - \int u\cdot\nabla\frac{|\nabla\theta|^2}{2}\\ &= -\int (\nabla u\cdot\nabla\theta)\cdot\nabla\theta \leqq \|\nabla u\|_{L^\infty}\,\|\nabla\theta(t)\|^2.\end{aligned}$$

We deduce

$$\frac{1}{2}\partial_t \left(t\,\|\nabla\theta(t)\|^2\right) + \nu\, t\, \|\Delta\theta(t)\|^2 \leqq \|\nabla u\|_{L^\infty}\, t\, \|\nabla\theta(t)\|^2 + \|\nabla\theta(t)\|^2. \tag{188}$$

By appropriate linear combination of (186), (187), (188), we get

$$\partial_t \left(t\,\|\nabla\theta(t)\|^2\right) + \frac{1}{\nu}\partial_t\|\theta(t)\|^2 + \frac{\|\nabla u\|_{L^\infty}}{\nu}\partial_t\left(t\,\|\theta(t)\|^2\right) \leqq C\,\frac{\|\nabla u\|_{L^\infty}}{\nu}\,\|\theta_0\|^2.$$

An integration in time gives

$$\|\nabla\theta(t)\|^2 \leqq \frac{1}{\nu t}\|\theta_0\|^2 + \frac{\|\nabla u\|_{L^\infty}}{\nu}\,\|\theta_0\|^2$$

and hence (185) for $p = 2$ follows.

To treat the general case, we shall use the results of article [13] on Green functions of advection–diffusion equations. First, we write (184) as

$$\partial_t\theta + u\cdot\nabla\theta - \Delta\theta + A\theta = -A\theta,$$

for some $A = A(\|\nabla u\|_{L^\infty})$, which will be chosen sufficiently large but independent of ν. By Duhamel's formula,

$$S(t)\theta_0 = \theta(t) = S_A(t)\theta_0 - A\int_0^t S_A(t-s)\theta(s)\,\mathrm{d}s,$$

where S_A is the semigroup associated to $\partial_t + u\cdot\nabla - \nu\Delta + A$. It is thus enough to derive the following bound for S_A:

$$\|\nabla S_A(t)\|_{\mathcal{L}(L^p)} \leqq \frac{C}{\sqrt{\nu}}\,\frac{\mathrm{e}^{-t}}{\sqrt{t}}. \tag{189}$$

Indeed from this estimate and Duhamel formula, we get

$$\|\nabla\theta(t)\|_p \leqq \frac{C}{\sqrt{\nu t}}\mathrm{e}^{-t}\|\theta_0\|_p + \int_0^t \frac{C}{\sqrt{\nu(t-s)}}\mathrm{e}^{-(t-s)}\,\|\theta(s)\|_p\,\mathrm{d}s$$

and since $\|\theta(s)\|_p \leqq \|\theta_0\|_p$ (note that this is here that we use that u is divergence free), we finally get

$$\|\nabla\theta(t)\|_p \leqq \frac{C}{\sqrt{\nu}}\left(\frac{1}{\sqrt{t}}+1\right)\|\theta_0\|_p$$

and hence (185) is proven.

To prove (189), we first notice that by standard energy estimates on θ and $\nabla\theta$, one has for $A = A(\|\nabla u\|_{L^\infty})$ large enough and independent of ν,

$$\|S_A(t)\|_{\mathcal{L}(W^{1,p})} \;\leqq\; C\mathrm{e}^{-t}.$$

By using this estimate and the semigroup property, we see that it is sufficient to show that, for some $T > 0$ small but independent of ν, we have for $t \leqq T$,

$$\|\nabla S_A(t)\|_{\mathcal{L}(L^p)} \;\leqq\; \frac{C}{\sqrt{\nu t}}. \tag{190}$$

Now, this bound can be obtained as a consequence of the work [13] on Green functions for advection diffusion operators. Let G_A the Green function satisfying

$$S_A(t)\theta_0(y) \;=\; \int_{\mathcal{R}^2} G_A(t,y,z)u_0(z)\,\mathrm{d}z.$$

Following [13], one shows that, for small time, the Green function satisfies for some small $T > 0$ independent of ν,

$$\sup_{t\leqq T,y}\int_{\mathbb{R}^2}|G_A(t,y,z)|\,\mathrm{d}z \;+\; \sup_{t\leq T,y}\sqrt{\nu t}\int_{\mathbb{R}^2}|\nabla_y G_A(t,y,z)|\,\mathrm{d}z \;\leq\; C,$$

$$\sup_{t\leqq T,z}\int_{\mathbb{R}^2}|G_A(t,y,z)|\,\mathrm{d}y \;+\; \sup_{t\leq T,z}\sqrt{\nu t}\int_{\mathbb{R}^2}|\nabla_y G_A(t,y,z)|\,\mathrm{d}y \;\leq\; C.$$

Indeed, we can use Section 2 of [13], where a general construction of Green functions for small-viscosity parabolic equations starting from a sufficiently accurate approximate Green function is derived. In Section 3 of [13] an application to one-dimensional parabolic systems in the whole space is given starting from an approximate Green function made by heat kernels moving along characteristics. In the multidimensional scalar framework, we can use exactly the same method, the choice

$$G^{\mathrm{app}}(t,x,y) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}}\exp\left(-\frac{|x-X(t-\tau,y)|^2}{4\nu t}\right),$$

where X is defined as the solution of

$$\partial_t X = u(X), \quad X(0,y) = y$$

gives the result.

Estimate (190) follows easily from the Green function estimates. $\square$

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