Roughness-induced effects on the quasi-geostrophic model

Didier Bresch, *David Gérard-Varet †

We study in this paper the effect of small-scale irregularities on the quasi-geostrophic model. This study is motivated by some problems related to oceanography, as the Gulf Stream separation, or the impact of the topography on the global circulation.

We first consider the role of coastal roughness in the phenomenon of western intensification of boundary currents. We show that the roughness is responsible for a nonlinear dynamics of the boundary layers, governed by a quasilinear elliptic equation. We thus extend substantially the classical derivation of Munk layers [15] and the results of convergence obtained in [10].

We then discuss the effect of a rough topography, by generalizing and justifying some formal computations of [17]. In particular, we derive rigorously a simplified model of oceanic circulation, with a nonlinear and nonlocal dissipative term due to the roughness.

1 Introduction

This paper is devoted to the study of the so-called "quasi-geostrophic" system. We shall derive rigorously some new asymptotic models, accounting for roughness-induced effects. Before we present our results in more detail, let us introduce the problem in a precise way.

The quasi-geostrophic system is a classical model of oceanic circulation. It reads

$$\left\{\begin{aligned} (\partial_t + u_1\partial_x + u_2\partial_y)\,(\Delta\Psi + \beta y - F\Psi + \eta_B) + r\Delta\Psi = \beta\,\mathrm{curl}\,\tau + \mathrm{Re}^{-1}\Delta^2\Psi,\\ u = (u_1, u_2)^t = \nabla^\perp\Psi,\\ \Psi|_{\partial\Omega} = \frac{\partial\Psi}{\partial n}|_{\partial\Omega} = 0,\\ \Psi|_{t=0} = 0. \end{aligned}\right. \tag{1}$$

In these equations :

- $\Psi = \Psi(t, \mathrm{x}) \in \mathbb{R}$ is a stream function, associated to the velocity field $u = (u_1(t, \mathrm{x}), u_2(t, \mathrm{x}))^t$. The time variable t lies in $\mathbb{R}^+$, and the space variable $\mathrm{x} = (x, y)$ lies in a two-dimensional domain Ω to be precised later on.
- $d/dt = \partial_t + u_1\partial_x + u_2\partial_y$ is the transport operator by the two-dimensional flow.
- $\Delta\Psi$ is the vorticity, and $r\Delta\Psi$, $r > 0$, is the Ekman pumping term.

*L.M.C., U.M.R. 5523, Université Joseph Fourier, 38041 Grenoble
†U.M.P.A, U.M.R. 5669, E.N.S. Lyon, 46 Allée d'Italie, 69364 Lyon

- $F\Psi$ is due to the free surface with F the Froude number.
- βy is the second term in the development of the Coriolis force.
- η_B is a bottom topography term.
- $\beta \operatorname{curl} \tau$ is the vorticity created by the wind, where τ is a given stress tensor.
- $\mathrm{Re}^{-1}\Delta^2\Psi$ is the usual viscosity term.

Formally, system (1) is derived starting from three dimensional Navier-Stokes equations with a Coriolis force and a free surface : see [15] or [8] for a brief recall of this derivation. Mathematically, the justification of this derivation is open. But if the free surface is replaced by a rigid lid approximation, (1) with $F = 0$ can be rigorously derived, as shown by B. Desjardins and E. Grenier in [10]. Note that the derivation of the Quasi-geostrophic equations with $F \neq 0$ may be mathematically proved starting from a viscous Shallow water equations, see for instance [5].

Despite its simplicity, this model catches some of the main features of oceanic circulation. We refer the reader to [3], [15] for an extensive physical discussion. From a mathematical viewpoint, qualitative properties of(1) have also been widely studied [7, 10, 1, 4, 6]. Among the related papers, let us mention the important work of B. Desjardins and E. Grenier [10]. These authors have performed a complete boundary layer analysis, under various asymptotics, in domains

$$\Omega = \{\chi_w(y) \leq x \leq \chi_e(y), \quad y_{min} \leq y \leq y_{max}\}\,.$$

They have notably derived the so-called Munk layers, responsible for the western intensification of boundary currents. In their stationary version, equations (1) have been studied in [1, 4, 6], looking at for instance the influence of an island or the influence of the choice of boundary conditions. Remark that there exist other quasi-geostrophic models, see [9], depending on the small scale parametrizations.

The aim of the present paper is to study roughness-induced effects on the quasi-geostrophic model. This work has strong physical motivation: the ocean bottom and coastal topography vary over a wide range of scales, the smallest of which can not be resolved in numerical computations. Such small-scale variations can not be seen in former theoretical studies, as topography terms depend regularly on x. There is therefore a need for the development of simplified models that account for the impact of the roughness on the large-scale ocean circulation.

In a first part, we investigate the influence of rough shores. We consider the asymptotic behaviour of solutions Ψ of (1), as β goes to infinity, in a domain with rough boundaries. We extend the convergence result obtained in [10] about Munk layers. In the rough case, we show that the dynamics of the boundary layers is strongly modified. The usual Munk system of ordinary differential equations is replaced by a quasilinear elliptic equation, leading to serious additional difficulties. To overcome these difficulties, we use some methods introduced by one of the authors in [12], for rotating fluids in rough domains.

In our view, it is an important step to progress in the understanding of the so-called "Gulf stream separation". This expression refers to the abrupt separation of the Gulf Stream from the North American coastline, at Cape Hatteras. This phenomenon, which has been observed for many years, seems to have very little variability: the current leaves the continent on a straight path without any visible deflection at the separation point. However, it is still poorly understood from a physical point of view: see [2] for more physical insight (in particular for a numerical study of the influence of boundary conditions on the separation point). Therefore, it is important to try to obtain new models which reflect the Gulf stream separation, taking into account for instance the effect of rough boundaries.

In a second part, we investigate the effect of a rough bottom topography. We consider rapidly oscillating functions $\eta_B^\zeta = \eta_B(x, y, x/\zeta, y/\zeta)$, periodic in both variables, with ζ a small positive parameter. Such a study has been carried at a formal level and in the linear regime in articles [17] and [18]. We consider in this paper the "weakly nonlinear" case, which seems to be the most relevant (see [17] for a discussion on the choice of the scalings). That means:

$$\zeta \to 0, \quad \Psi = \zeta\psi, \quad u = \zeta v, \quad \mathrm{Re}^{-1} = \nu\zeta^2, \quad \beta,\, r,\, \nu > 0 \text{ given.}$$

Thus, system (1) becomes

$$\begin{cases} (\partial_t + \zeta v \cdot \nabla)\,(\Delta\psi + \dfrac{\beta y}{\zeta} + \dfrac{1}{\zeta}\eta_B^\zeta) + r\Delta\psi = \beta \operatorname{curl}\tau + \nu\zeta^2\Delta^2\psi \\ \hfill v = (v_1, v_2)^t = \nabla^\perp\psi, \\ \hfill \psi|_{t=0} = 0. \end{cases} \tag{2}$$

We show convergence results on the solutions ψ^ζ of (2) as $\zeta \to 0$. As will be seen in the sequel, the limit system has a dissipative term, due to the small-scale roughness. This term is both nonlinear and nonlocal. In the linear case, it degenerates into a convolution product, turning the limit system into an integro differential system. All these results are in agreement with formal computations of [17].

The plan of the paper is as follows. In Section 2, we describe precisely the different domains we will consider and state the main results of the paper. In Section 3, we analyze the case of rough shores. In Section 4, we look at the case of the periodic topography η_B^ζ.

2 Statement of the results

2.1 Rough shores

In order to lighten notations, we assume throughout the study of rough coasts (section 3) that

$$r = 1, \qquad \mathrm{Re} = 1.$$

Up to minor changes, similar results would hold for arbitrary constants r and Re. Moreover, we use the parameter $\varepsilon = \beta^{-1/3}$ preferentially to β. The reason is that ε is the natural size of the boundary layers arising in this study. Assuming $F = 0$, System (1) reads then

$$\begin{cases} (\partial_t + u_1\partial_x + u_2\partial_y)\,(\Delta\Psi + \varepsilon^{-3}y + \eta_B) + \Delta\Psi = \varepsilon^{-3}\operatorname{curl}\tau + \Delta^2\Psi, \\ \hfill u = (u_1, u_2)^t = \nabla^\perp\Psi, \\ \hfill \Psi|_{\partial\Omega} = \dfrac{\partial\Psi}{\partial n}|_{\partial\Omega} = 0, \\ \hfill \Psi|_{t=0} = 0. \end{cases} \tag{3}$$

2.1.1 The rough domain Ω^ε

Let us describe the domain Ω^ε where Equations (3) hold (see also figure 2.1.1). We write

$$\Omega^\varepsilon = \Omega^\varepsilon_w \cup \Sigma_w \cup \Omega \cup \Sigma_e \cup \Omega^\varepsilon_e.$$

- Ω is the "interior domain", which, following notations of [10], is defined by

  $$\Omega = \Big\{ \chi_w(y) \le x \le \chi_e(y), \quad y \in [y_{min}, y_{max}] \Big\},$$

  where χ_w and χ_e are smooth functions defined for $y \in [y_{min}, y_{max}]$.

- $\Sigma_w = \Big\{ (\chi_w(y), y), \ y \in (y_{min}, y_{max}) \Big\}$ and $\Sigma_e = \Big\{ (\chi_e(y), y), \ y \in (y_{min}, y_{max}) \Big\}$ are "interfaces".

- Ω^ε_w and Ω^ε_e are the "rough shores". More precisely, let $\gamma_w = \gamma_w(Y)$ and $\gamma_e = \gamma_e(Y)$ smooth, positive and 1-periodic functions. We set

$$\Omega^\varepsilon_w = \Big\{ (x,y), \quad 0 > x - \chi_w(y) > \ -\varepsilon \gamma_w(\varepsilon^{-1} y) \Big\}, \tag{4}$$

$$\Omega^\varepsilon_e = \Big\{ (x,y), \quad 0 < x - \chi_e(y) < \ \varepsilon \gamma_e(\varepsilon^{-1} y) \Big\}. \tag{5}$$

We also define lateral boundaries

$$\Gamma^\varepsilon_w = \Big\{ \big(\chi_w(y) - \varepsilon \gamma_w(\varepsilon^{-1} y), y \big) \, , \ y \in (y_{min}, y_{max}) \Big\},$$

$$\Gamma^\varepsilon_e = \Big\{ \big(\chi_e(y) + \varepsilon \gamma_e(\varepsilon^{-1} y), y \big) \, , \ y \in (y_{min}, y_{max}) \Big\}.$$

Remark: Up to a few more technicalities, one could consider more general (hence more realistic) domains: results below extend to functions $\gamma_w = \gamma_w(y, Y)$ (resp. $\gamma_e = \gamma_e(y, Y)$), with $\gamma_w(y, \cdot)$ $T_w(y)$-periodic (resp. $\gamma_e(y, \cdot)$ $T_e(y)$-periodic).

2.1.2 The boundary layer domains

The study of the boundary layers requires additional boundary layer domains (figure 2.1.2). Namely, we define $\omega_w = \omega_w^+ \cup \sigma_w \cup \omega_w^-$, where

$$\omega_w^+ = \Big\{ X > 0, \quad Y \in (0,1) \Big\}, \qquad \sigma_w = \Big\{ X = 0, \quad Y \in (0,1) \Big\},$$

$$\omega_w^- = \Big\{ (X,Y), \quad Y \in (0,1), \quad -\gamma_w(Y) < X < 0 \Big\}.$$

We define similarly $\omega_e = \omega_e^- \cup \sigma_e \cup \omega_e^+$. We call n_w (resp. n_e) the outward unit normal vector at the boundary $\{X = -\gamma_w(Y)\}$ (resp. $\{X = \gamma_e(Y)\}$).

For all $R > 0$, we denote

$$\omega^R = \omega_w \cap \{X > R\}.$$

Finally, we set $\Omega_w = \cup_{k \in \mathbb{Z}} \Big\{ (X, Y+k), \quad X > -\gamma_w(Y), \quad Y \in [0,1] \Big\}$

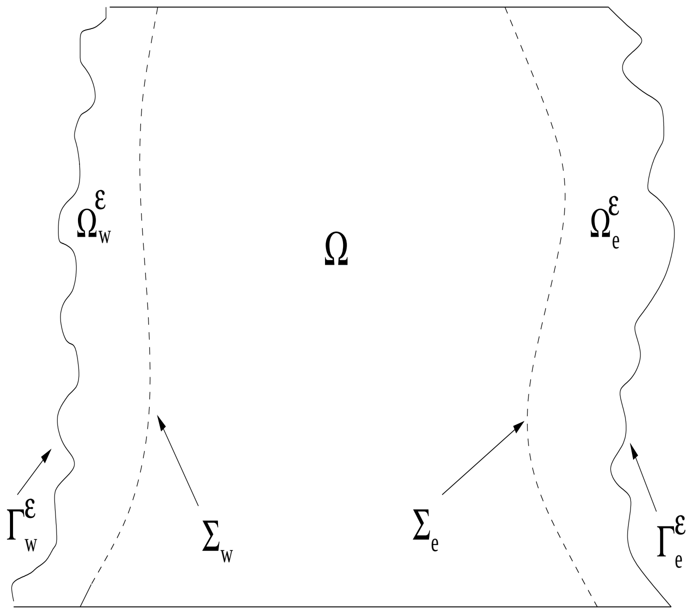

Figure 1: The rough domain Ω^ε.

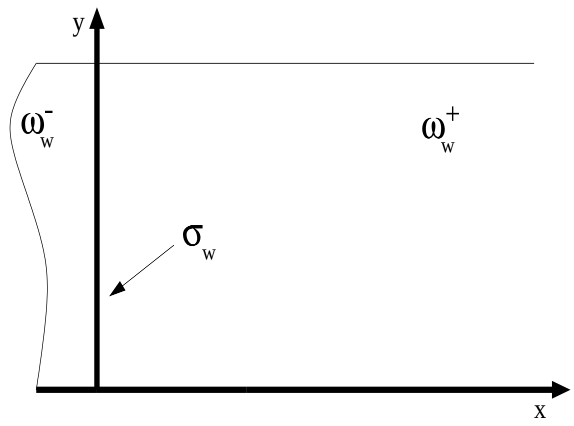

Figure 2: The boundary layer domain ω_w.

2.1.3 Other notations

Let us introduce some more notations that will be useful in the study of the western boundary layer.

Operators

For all $y \in [y_{min}, y_{max}]$, we set

$$\nabla_w(y) = (\partial_X, \partial_Y - \chi_w'(y)\partial_X)^t, \quad \nabla_w^\perp(y) = (\chi_w'(y)\partial_X - \partial_Y, \partial_X)^t,$$

and define the operators $\Delta_w(y)$ and $Q_w(y)$ by:

$$\Delta_w(y) = \nabla_w(y)^2 = \partial_X^2 \ + \ (\chi_w'(y)\partial_X - \partial_Y)^2,$$

$$Q_w(y)(\Psi, \widetilde{\Psi}) = \nabla_w^\perp(y) \cdot \left(\left(\nabla_w^\perp(y)\Psi \cdot \nabla_w(y)\right) \cdot \nabla_w^\perp(y)\widetilde{\Psi} \right).$$

Similarly,

$$\nabla_e(y) = (\partial_X, -\chi_e'(y)\partial_X - \partial_Y)^t, \quad \nabla_e^\perp(y) = (\chi_e'(y)\partial_X + \partial_Y, \partial_X)^t,$$

and

$$\Delta_e(y) = \nabla_e(y)^2 = \partial_X^2 \ + \ (\chi_e'(y)\partial_X + \partial_Y)^2,$$

$$Q_e(y)(\psi, \tilde{\psi}) = \nabla_e^\perp(y) \cdot \left(\left(\nabla_e^\perp(y)\psi \cdot \nabla_e(y)\right) \cdot \nabla_e^\perp(y) \right).$$

Functional spaces

We name

$$\mathcal{V} = \Big\{ \varphi \in \mathcal{C}^\infty(\Omega_w), \quad \varphi \text{ 1-periodic in } Y \Big\}, \tag{6}$$

$$\mathcal{V}_0 = \Big\{ \varphi \in \mathcal{V}, \quad (\operatorname{supp}\varphi) \cap \partial\Omega_w = \emptyset, \quad \operatorname{supp}\varphi \text{ bounded in the X direction} \Big\}. \tag{7}$$

Finally, we define

$$\mathcal{H}^2(\omega_w) = \text{ the closure of } \mathcal{V} \text{ in } H^2(\omega_w), \tag{8}$$

$$\mathcal{H}_0^2(\omega_w) = \text{ the closure of } \mathcal{V}_0 \text{ in } H^2(\omega_w), \tag{9}$$

$$\mathcal{D}_0^2(\omega_w) = \text{ the completion of } \mathcal{V}_0 \text{ for the norm } \|\psi\| = \|D^2\psi\|_{(L^2(\omega_w))^4}, \tag{10}$$

where $D^2\psi$ is the Hessian matrix of ψ.

2.1.4 Results

We want to consider System (3) in the limit of small ε, in the domain Ω^ε defined above. Let $T > 0$, for the sake of simplicity we assume that

$$\operatorname{curl}\tau \in \mathcal{C}^\infty\left([0,T] \times \overline{\Omega^\varepsilon}\right), \quad \operatorname{curl}\tau|_{t=0} = 0.$$

Since we consider a problem with zero initial data and a source term, we are actually studying well-prepared data, as in [10]. Moreover, in order to avoid problems near northern and southern boundaries, we assume as in [10] that:

$$\exists\, \lambda > 0, \quad \operatorname{curl}\tau = 0 \quad \text{for } \mathrm{y} \in [\mathrm{y}_{\max} - \lambda, \mathrm{y}_{\max}] \cup [\mathrm{y}_{\min}, \mathrm{y}_{\min} + \lambda]\,. \tag{11}$$

As pointed out in [10], System (3) is the vorticity formulation of a two-dimensional Navier-Stokes type equation : for all $\varepsilon > 0$, (3) has a unique smooth solution Ψ^ε.

As usual in boundary layer problems, the study of Ψ^ε will require an additional system. For $t \in [0,T]$, x in Ω^ε, let

$$\Psi_{int}(t,\mathrm{x}) := \int_{\chi_e(y)}^{x} \operatorname{curl}\tau(t,x',y)\,dx', \tag{12}$$

and for $t \in [0,T]$, $y \in [y_{min}, y_{max}]$,

$$\phi(t,y) := \int_{\chi_e(y)}^{\chi_w(y)} \operatorname{curl}\tau(t,x',y)\,dx'. \tag{13}$$

For all $t \in [0,T]$ and $y \in [y_{min}, y_{max}]$, we consider the following boundary layer systems, of unknown $\Psi^{t,y} = \Psi^{t,y}(\mathrm{X})$, for $\mathrm{X} \in \omega_w$:

$$Q_w(y)\left(\Psi^{t,y}, \Psi^{t,y}\right) + \partial_X \Psi^{t,y} - \left(\Delta_w(y)\right)^2 \Psi^{t,y} = 0, \tag{14}$$

with the boundary conditions :

$$\begin{cases} \Psi^{t,y} \ \text{ 1-periodic in } \ Y, \\ \Psi^{t,y}|_{X=-\gamma_w(Y)} = -\phi(t,y), \quad \dfrac{\partial \Psi^{t,y}}{\partial n_w}|_{X=-\gamma_w(Y)} = 0. \end{cases} \tag{15}$$

Note that for fixed y, $\Delta_w(y)$ (and consequently $\left(\Delta_w(y)\right)^2$) is an elliptic operator. We prove in Section 3 the following

Theorem 2.1 *There exist a constant $\phi_\infty > 0$ and a function*

$$\Psi_w : [0,T] \times [y_{min}, y_{max}] \times \omega_w \mapsto \mathbb{R},$$

such that if $\|\phi\|_\infty < \phi_\infty$, for all (t,y), $\Psi^{t,y} = \Psi_w(t,y,\cdot)$ is the unique weak solution in $\mathcal{H}^2(\omega_w)$ of (14)-(15) (see remark below).
Moreover, $\Psi^{t,y}$ belongs to $H^m(\omega_w)$ for all $m \geq 0$, and satisfies the estimate

$$\sup_{t,y} \|\Psi_w(t,y,\cdot)\|_{H^m(\omega^R)} \ \leq\ C_m \exp(-\sigma R), \quad R \geq R_1, \tag{16}$$

where σ and R_1 are independent of m.

Remark: For all (t, y) and for all $\Psi^{t,y}$ in $\mathcal{H}^2(\omega_w)$, one can easily see that equation (14) has a meaning in $\left(\mathcal{H}^2_0(\omega_w)\right)'$. Furthermore, the boundary conditions (15) have also a meaning in the trace sense. Hence, "the unique weak solution of (14)-(15)" means the only function $\Psi^{t,y}$ satisfying (14) in $(\mathcal{H}^2_0(\omega_w))'$, and (15) in the trace sense.

Once this auxiliary system is solved, we can prove our main convergence result. We prove in Section 3

Theorem 2.2 *Let Ψ^ε be the solution of System (3). There exists C_∞, such that if $\|\phi\|_\infty < C_\infty$, then*

$$\|\Psi^\varepsilon - \Psi^\varepsilon_{\text{app}}\|_{L^\infty(0,T;H^1(\Omega^\varepsilon))} \to 0 \ \textit{as}\ \varepsilon \to 0,$$

where

$$\Psi^\varepsilon_{app}(t, \mathrm{x}) = \Psi_{int}(t, \mathrm{x}) + \Psi_w\left(t, y, \frac{x - \chi_w(y)}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

where Ψ_{int} is given by (12) and Ψ_w is given by Theorem 2.1.

Remark: The previous theorem provides an approximation of Ψ^ε at the main order. In fact for the proof, we will have to build an approximation up to the order ε that means under the form

$$\Psi^\varepsilon_{app} = \Psi_{int}(t, \mathrm{x}) + \Psi^0_w\left(t, y, \frac{x - \chi_w(y)}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon\left(\Psi^0_e\left(t, y, \frac{x - \chi_w(y)}{\varepsilon}, \frac{y}{\varepsilon}\right) + \Psi^1_w\left(t, y, \frac{x - \chi_w(y)}{\varepsilon}, \frac{y}{\varepsilon}\right)\right).$$

The two last quantities being of size $\sqrt{\varepsilon}$ in the H^1 norm, Theorem 2.2. will be obtained by a classical energy estimates on the difference $\Psi^\varepsilon - \Psi^\varepsilon_{app}$.

Remark: Theorem 2.2 is the generalization to rough boundaries of the convergence theorem obtained in [10], for a domain Ω without roughness. They proved a similar asymptotic result, with an asymptotic behaviour

$$\Psi^\varepsilon_{app}(t, \mathrm{x}) = \Psi_{int}(t, \mathrm{x}) + \Psi_m\left(t, y, \frac{x - \chi_w(y)}{\varepsilon}\right).$$

The stream function Ψ_{int} was also given by (12), whereas the Munk layer term $\Psi_m = \Psi_m(t, y, X)$ was the solution of

$$\begin{cases} \partial_X \Psi_m(t, y, \cdot) \ - \ (1 + \chi_w'(y)^2)^2\ \partial_X^4 \Psi_m(t, y, \cdot) = 0, \\ \Psi_m(t, y, \cdot)|_{X=0} = -\phi(t, y), \qquad \partial_X \Psi_m(t, y, \cdot)\big|_{X=0} = 0, \end{cases} \tag{17}$$

with ϕ given by (13). Note that (17) is a simple linear differential equation in variable X. It is strongly different from the rough case, in which the boundary layer system (14)-(15) is genuinely two-dimensional, and nonlinear.

Remark: A smallness assumption on the function ϕ of (13) is crucial, as can be seen from both theorems. It is first needed to prove the existence of the boundary layer term Ψ_w. We then use it to prove the convergence theorem. Note that such condition was already present in the convergence theorem of [10]. As explained in this paper, it is deeply linked to the stability of the boundary layer : it can be read as a smallness condition on an appropriate Reynolds number.

Remark: We recover the fact that the velocity is very large ($O(\varepsilon^{-1/2})$ in L^2 norm), due to the western boundary layer: it is the so-called intensification of western currents [3].

2.2 Rough topography

2.2.1 The topography η_B^ζ

Throughout section 4, we will study Equations (2), in $\Omega = \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Up to slight modifications, the case $\Omega = \mathbb{R}^2$ could be handled similarly. We still assume that $\operatorname{curl}\tau \in \mathcal{C}^\infty([0,T]\times\Omega)$, with $\operatorname{curl}\tau|_{t=0} = 0$. We assume that the bottom topography is given by

$$\eta_B^\zeta(\mathrm{x}) = \eta_B\left(\mathrm{x}, \frac{\mathrm{x}}{\zeta}\right).$$

with $\eta_B = \eta_B(\mathrm{x},\mathrm{X})$ smooth, 1-periodic in its variables. We assume ζ such that $1/\zeta$ is an integer.

2.2.2 Results

We want to consider System (2) in the limit of small ζ. There again, we need auxiliary systems.

Let $T > 0$, and $U = U(t,\mathrm{x}) \in \left(L^\infty\left((0,T)\times\Omega\right)\right)^2$. We first consider the following equations, of unknown $\tilde\psi = \tilde\psi(t,\mathrm{x},\mathrm{X})$ for $t > 0$ and $(\mathrm{x},\mathrm{X}) \in \mathbb{T}^2\times\mathbb{T}^2$:

$$\begin{cases} \partial_t\Delta_{\mathrm{X}}\tilde\psi + \tilde u\cdot\nabla_{\mathrm{X}}\left(\Delta_{\mathrm{X}}\tilde\psi\right) + U\cdot\nabla_{\mathrm{X}}\left(\Delta_{\mathrm{X}}\tilde\psi\right) + \tilde u\cdot\nabla_{\mathrm{X}}\eta_B + r\Delta_{\mathrm{X}}\tilde\psi \\ \qquad\qquad -\nu\,(\Delta_{\mathrm{X}})^2\tilde\psi + U\cdot\nabla_{\mathrm{X}}\eta_B = 0, \\ \tilde u = \nabla_{\mathrm{X}}^\perp\tilde\psi = \left(-\partial_Y\tilde\psi, \partial_X\tilde\psi\right), \quad \tilde\psi|_{t=0} = 0. \end{cases} \tag{18}$$

Theorem 2.3 *There exists a unique weak solution $\widetilde\psi$ of* (18),

$$\widetilde\psi \in L^\infty\left(0,T\times\Omega;\ H^1(Q)\right)\cap L^2\left(0,T;\ L^\infty(\Omega;\ H^2(Q))\right), \int_{\mathbb{T}^2}\widetilde\psi(t,\cdot) = 0.$$

We now introduce what will be shown to be the limit system of (2). First, on the basis of Theorem 18, we define $\mathcal{F} : (L^\infty((0,T)\times\Omega))^2 \mapsto (L^\infty((0,T)\times\Omega))^2$ by: for all $\mathrm{x}\in\mathbb{T}^2$,

$$\mathcal{F}(U)(t,\mathrm{x}) = -\int_{\mathbb{T}^2_{\mathrm{X}}}\eta_B(\mathrm{x},\mathrm{X})\,\nabla_{\mathrm{X}}\tilde\psi(t,\mathrm{x},\mathrm{X})\,d\mathrm{X},$$

where $\tilde\psi$ is the solution of (18), given by theorem 2.3. We will show that it is dissipative, in the sense of

Proposition 2.4 *For all U in $(L^\infty((0,T)\times\Omega))^2$, for all $t\in(0,T)$, for all* x,

$$\int_0^t \mathcal{F}(U)(s,\mathrm{x})\cdot U(s,\mathrm{x})\,ds \ge 0.$$

The limit system is: for $t > 0$, $\mathrm{x}\in\mathbb{T}^2_{\mathrm{x}}$,

$$\begin{cases} \partial_t\Delta_{\mathrm{x}}\psi + \beta\partial_x\psi + u\cdot\nabla_{\mathrm{x}}\overline{\eta_B} + r\Delta_{\mathrm{x}}\psi + \operatorname{curl}_{\mathrm{x}}\mathcal{F}(u) = \beta\operatorname{curl}\tau, \\ u = \nabla_{\mathrm{x}}^\perp\psi, \\ \psi|_{t=0} = 0, \end{cases} \tag{19}$$

where the bar stands for the average in the rough variable X. We show in Section 4:

Theorem 2.5 *For all $m > 1$, there exists $T_m > 0$, such that System* (19) *has a unique solution*

$$\psi \in \mathcal{C}\left([0,T_m]; H^{m+1}(\mathbb{T}^2)\right),$$

with for all $t \geq 0$, $\int_{\mathbb{T}^2} \psi(t,x)\, d\mathrm{x} = 0$.

Remark: The dissipative term $\mathrm{curl}_{\mathrm{x}}\, \mathcal{F}(u^0)$ in the limit system is of course due to the roughness: it is the mathematical translation of energy loss due to the friction at the bottom.

Remark: System (19) is close to the one derived in [12], for rotating fluids in rough domains. Namely, the limit equation was of type

$$\partial_t \Delta_{\mathrm{x}} \psi \; + \; u \cdot \nabla_{\mathrm{x}} \Delta_{\mathrm{x}} \psi + \mathrm{curl}_{\mathrm{x}}\; \mathcal{F}(u) = 0.$$

for a function $\mathcal{F}$ defined in a neighborhood of 0 in $\mathbb{R}^2$, with values in $\mathbb{R}^2$. In particular, in both cases, we do not manage to control $\mathrm{curl}_{\mathrm{x}}\, \mathcal{F}(u)$ in L^p spaces, so that we can not conclude to the existence of global in time solutions, for instance through a Yudovitch scheme (cf [14]). However, we wish to point out a strong difference between the two cases: the function $\mathcal{F}$ is here non-local in time, whereas it was local for the rotating fluids system. In other words, for our model of oceanic circulation, the dissipative mechanisms depend on the whole history of the large scale flow. This appears clearly when one considers the linear case, i.e. omitting the nonlinear term in (2). Following [17], Equation (19) becomes

$$\partial_t \Delta_{\mathrm{x}} \psi \; + \; \beta \partial_x \psi \; + \; u \cdot \nabla_{\mathrm{x}} \overline{\eta_B} \; + \; r \Delta_{\mathrm{x}} \psi \; + \; \mathrm{curl}_{\mathrm{x}} \int_0^t R(t-\tau, \cdot) u(\tau, \cdot)\, d\tau = \beta \; \mathrm{curl}\, \tau, \qquad (20)$$

for some kernel R. Thus, ψ obeys an integro-differential rather than a differential equation. For more on this linear case (such as numerical computations), see [17].

We now state the convergence result on solutions ψ^ζ of System (2):

Theorem 2.6 *Let $m > 1$. Let $u = \nabla_{\mathrm{x}}^{\perp} \psi$, where $\psi \in \mathcal{C}\left([0,T_m]; H^{m+1}(\mathbb{T}^2_{\mathrm{x}})\right)$ is the solution of* (19) *given by theorem 2.5. Let $u^\zeta = \nabla_{\mathrm{x}}^{\perp} \psi^\zeta$, where ψ^ζ is the solution of* (2)*. Then*

$$u^\zeta \to u \;\; in \;\; \left(L^\infty\left((0,T_m); L^2(\Omega)\right)\right)^2, \quad as\; \zeta \to 0.$$

3 Rough coasts

This section is devoted to the proof of Theorems 2.1 and 2.2. In subsection 3.1, we formally derive an asymptotic expansion of the solution Ψ^ε of (3). We recover formally functions Ψ_{int} and Ψ_w defined in the previous section. In subsection 3.2, we prove Theorem 2.1. In subsection 3.3, we prove Theorem 2.2.

3.1 Formal asymptotic expansion

In this part, we build formally an approximate solution Ψ^ε_{app} of System (3).

3.1.1 Ansatz

We look for an approximation of the following type

$$\Psi^\varepsilon_{\text{app}}(t,x,y) = \sum_{i=0}^{n} \varepsilon^i \left(\Psi^i(t,x,y) \;+\; \Psi^i_w\left(t,y,\frac{x-\chi_w(y)}{\varepsilon},\frac{y}{\varepsilon}\right) \;+\; \Psi^i_e\left(t,y,\frac{\chi_e(y)-x}{\varepsilon},\frac{y}{\varepsilon}\right)\right) \quad (21)$$

where

- $\Psi^i(t,x,y)$ are interior terms, defined for $t>0$, $(x,y)\in\Omega^\varepsilon$, with support in the interior domain $\overline{\Omega}$.
- $\Psi^i_w = \Psi^i_w(t,y,X,Y)$ and $\Psi^i_e = \Psi^i_e(t,y,X,Y)$ are respectively the western and eastern boundary layer profiles. They are defined for $t>0$, and X respectively in ω_w and ω_e, with 1-periodicity in the Y variable.

Dirichlet conditions

We expect the boundary layer terms to have no role far from the boundaries, which leads to the condition

$$\Psi^i_e,\ \Psi^i_w \xrightarrow[X\to\infty]{} 0. \quad (22)$$

We then want our approximate solution to satisfy $\Psi^\varepsilon_{app}=0$ at $\partial\Omega^\varepsilon$. It yields the following Dirichlet conditions: for all (t,y),

$$\Psi^i_w(t,y,\cdot)\Big|_{X=-\gamma_w(Y)} = 0, \quad \Psi^i_e(t,y,\cdot)\Big|_{X=\gamma_e(Y)} = 0. \quad (23)$$

Neumann conditions

We then want that $\partial_n \Psi^\varepsilon_{app}=0$ at $\partial\Omega^\varepsilon$. It yields easily Neumann conditions on the boundary layer profiles, of the type: for all (t,y),

$$\frac{\partial\Psi^i_w}{\partial n_w}\Big|_{X=-\gamma_w(Y)} = \varphi^i_w(t,y,X)|_{X=-\gamma_w(Y)}, \quad \frac{\partial\Psi^i_e}{\partial n_e}\Big|_{X=-\gamma_e(Y)} = \varphi^i_e(t,y,X)|_{X=-\gamma_e(Y)}, \quad (24)$$

where functions φ^i_w and φ^i_e involve the Ψ^k_w 's and Ψ^k_e for $k\le i-1$.

Interface conditions

It remains to determine conditions at the interfaces. As the interior terms are zero outside $\overline{\Omega}$, they create discontinuities at the interfaces Σ_w and Σ_e. Then boundary layer terms are added to cancel such discontinuities. This procedure is classical in roughness effect problems, see for instance [13], [12]. More precisely, to prove the convergence result, we will carry energy estimates on the difference of the stream functions $\phi^\varepsilon = \Psi^\varepsilon - \Psi^\varepsilon_{app}$. The roughness corrector Ψ^ε_{ext} is made to drop the surface integral that appears when evaluating the viscous term

$$\int_{\Omega^\varepsilon} (\Delta^2\phi^\varepsilon)\phi^\varepsilon = \int_{\Sigma_e\cup\Sigma_w} \left[\partial_n(\Delta\Psi^\varepsilon_{app})\Psi^\varepsilon_{app} \;-\; \nabla\partial_n\Psi^\varepsilon_{app}\cdot\nabla\Psi^\varepsilon_{app}\right]_{\Sigma_e\cup\Sigma_w} + \int_{\Omega^\varepsilon}|D^2\phi^\varepsilon|^2.$$

Sufficient conditions for this integral to vanish are

$$\left[\partial_x^k \Psi^\varepsilon_{app}\right]_{\Sigma_e\cup\Sigma_w} = 0, \quad k=0,\dots,3, \tag{25}$$

Straightforward computations give the following jump conditions on the boundary layer terms:

$$\left[\psi_w^i(t,y,\cdot)\right]_{|\sigma_w} = -\Psi^i(t,y,\cdot)|_{x=\chi_w(y)}, \quad \left[\psi_e^i(t,y,\cdot)\right]_{|\sigma_w} = -\Psi^i(t,y,\cdot)|_{x=\chi_e(y)}, \tag{26}$$

and

$$\left[\partial_X^k \psi_w^i(t,y,\cdot)\right]_{|\sigma_w} = f_w^{i,k}(t,y), \quad \left[\partial_X^k \psi_e^i(t,y,\cdot)\right]_{|\sigma_w} = f_e^{i,k}(t,y), \quad k=1,\dots,3 \tag{27}$$

where the $f_w^{i,k}$ and $f_e^{i,k}$, $k=1,\dots,3$, depend on the Ψ^j, Ψ_w^j and Ψ_e^j, $j\le i-1$.

3.1.2 Derivation of the first profiles

We now derive the equations satisfied by the first profiles of the expansion (21).

In the interior, we obtain at the leading order ε^{-3} the so-called Sverdrup relation

$$\partial_x \Psi^0 = \operatorname{curl}\tau \tag{28}$$

We choose Ψ^0 to cancel at Σ_e, so that

$$\begin{aligned}\Psi^0(t,x,y) &= \int_{\chi_e(y)}^x \operatorname{curl}\tau(t,x',y)\,dx' \quad \text{in } \Omega,\\ &= 0 \quad \text{in } \Omega^\varepsilon-\Omega.\end{aligned} \tag{29}$$

In the western layer, we have $\nabla\sim\varepsilon^{-1}\nabla_w(y)$. We find, for X in $\omega_w^+\cup\omega_w^-$:

$$Q_w(y)(\Psi_w^0(t,y,\cdot),\Psi_w^0(t,y,\cdot)) + \partial_X\Psi_w^0(t,y,\cdot) - (\Delta_w(y))^2\,\Psi_w^0(t,y,\cdot) = 0, \tag{30}$$

Remark: The quadratic operator Q_w is deduced from the quadratic term $(u\cdot\nabla)\Delta\Psi$ of system (3), replacing ∇ by $\nabla_w = (\partial_X, \partial_Y - \chi_w'\partial_X)^t$. Indeed, we remind that

$$(u\cdot\nabla)\Delta\Psi = (u\cdot\nabla)\operatorname{curl} u = \operatorname{curl}((u\cdot\nabla)u) = \nabla^\perp\cdot\Big(\left(\nabla^\perp\Psi\cdot\nabla\right)\nabla^\perp\Psi\Big)$$

We complete these equations by the interface and boundary conditions mentioned above, namely

$$\begin{cases}\left[\Psi_w^0(t,y,\cdot)\right]_{|\sigma_w} = -\phi(t,y),\\ \left[\partial_X^k\Psi_w^0(t,y,\cdot)\right]_{|\sigma_w} = 0, \quad k=1,\dots,3\\ \Psi_w^0(t,y,\cdot)\text{ 1-periodic in } Y, \quad \Psi_w^0(t,y,\cdot)\big|_{X=-\gamma_w(Y)} = \dfrac{\partial\Psi_w^0(t,y,\cdot)}{\partial n_w}\big|_{X=-\gamma_w(Y)} = 0.\end{cases} \tag{31}$$

In the eastern layer, we obtain homogeneous equations: for X $\in\omega_e^+\cup\omega_e^-$,

$$Q_e(y)(\Psi_e^0(t,y,\cdot),\Psi_e^0(t,y,\cdot)) - \partial_X\Psi_e^0(t,y,\cdot) - (\Delta_e(y))^2\,\Psi_e^0(t,y,\cdot) = 0. \tag{32}$$

The interface and boundary conditions are also homogeneous, thanks to our definition of Ψ^0:

$$\left[\partial_X^k \Psi_e^0(t,y,\cdot)\right]_{|\sigma_e} = 0, \qquad k=0,\dots,3, \tag{33}$$

$$\Psi_e^0(t,y,\cdot) \text{ 1-periodic in } Y, \quad \Psi_e^0(t,y,\cdot)\big|_{X=-\gamma_e(Y)} = \frac{\partial \Psi_e^0(t,y,\cdot)}{\partial n_e}\big|_{X=-\gamma_e(Y)} = 0, \tag{34}$$

Of course, this system has the solution $\Psi_e^0 \equiv 0$.

Remark: Our choice for the streamfunction Ψ^0 is the same as in [10]. It is due to the properties of the eastern coast, which cannot bear a large boundary layer. Indeed, as explained in [10], the equation satisfied by the eastern profile Ψ_e^0 in the non-rough case is

$$-\partial_X \Psi_e^0 - (1+(\chi_e')^2)^2\, \partial_X^4 \Psi_e^0 = 0. \tag{35}$$

The characteristic equation of this ODE has only one root with negative real part, so that one can not impose simultaneously conditions on $\Psi_e^0, \partial_x \Psi_e^0$ at the boundary *and* a condition at infinity.

The same difficulty appears in the rough case, as equations (32) "contain" the differential equation (35). Note that such difficulty is not present in the western layer, as the underlying ODE

$$\partial_X \Psi_w^0 - (1+(\chi_w')^2)^2\, \partial_X^4 \Psi_w^0 = 0.$$

has two characteristic roots with negative real parts.

This ends the derivation of the first corrector terms. If we gather differently the leading order terms of the approximation, we obtain formally that in Ω^ε,

$$\Psi_{app}^\varepsilon(t,\mathrm{x}) \sim \Psi_{int}(t,x,y) + \Psi_w\left(t, y, \frac{x-\chi_w(y)}{\varepsilon}, \frac{y}{\varepsilon}\right),$$

where $\Psi_w = \Psi_w(t,y,\mathrm{X})$ is defined by

$$\Psi_w(t,y,\mathrm{X}) = \Psi_w^0(t,y,\mathrm{X}), \quad \mathrm{X} \in \omega_w^+, \tag{36}$$

$$\Psi_w(t,y,\mathrm{X}) = \Psi_w^0(t,y,\mathrm{X}) - \Psi_{int}(t,x,y), \quad \mathrm{X} \in \omega_w^-. \tag{37}$$

Using equations (30)–(31), we get (still at a formal level) the following equations on $\Psi^{t,y} = \Psi_w(t,y,\cdot)$: for X in $\omega_w^+ \cup \omega_w^-$,

$$Q_w(y)(\Psi^{t,y},\Psi^{t,y}) + \partial_X \Psi^{t,y} - (\Delta_w(y))^2\, \Psi^{t,y} = 0, \tag{38}$$

with

$$\begin{cases} \left[\partial_X^k \Psi^{t,y}\right]_{|\sigma_w} = 0, \quad k=0,\dots,3, \\ \Psi^{t,y}\big|_{X=-\gamma_w(Y)} = -\phi(t,y), \quad \dfrac{\partial \Psi^{t,y}}{\partial n_w}\big|_{X=-\gamma_w(Y)} = 0. \end{cases} \tag{39}$$

As the jump conditions are homogeneous, equation (38) is in fact satisfied on all ω_w. We thus recover formally the boundary layer system and the asymptotic result of theorems 2.1 and 2.2.

3.2 The boundary layer system

The aim of this section is to prove theorem 2.1. To do so, we use an equivalent formulation of (14), (15), more appropriate for a variational treatment. Let us define $= \psi_m(t,y,\mathrm{X})$ by: for $\mathrm{X}=(X,Y)$ in ω_w^-, $\psi_m(t,y,\mathrm{X})=0$, and for $\mathrm{X}=(X,Y)$ in ω_w^+

$$\psi_m(t,y,\mathrm{X}) = -\phi(t,y)\left(\cos\left(\frac{\sqrt{3}}{2}D_w(y)X\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}D_w(y)X\right)\right), \tag{40}$$

where ϕ is given by (13), and $D_w(y) = (1+(\chi_w'(y))^2)^{-2/3}$. Note that for all (t,y), $\psi_m^{t,y}(\mathrm{X}) = \psi_m(t,y,\mathrm{X})$ satisfies (14) in $\omega_w^+ \cup \omega_w^-$, with the following jump conditions:

$$[\psi_m^{t,y}]|_{\sigma_w} = -\phi(t,y), \quad \left[\partial_X \psi_m^{t,y}\right]|_{\sigma_w} = 0.$$

If we set

$$\psi_w^0(t,y,\mathrm{X}) = \Psi_w(t,y,\mathrm{X}) - \psi_m(t,y,\mathrm{X}), \quad \mathrm{X}\in\omega_w^+, \tag{41}$$
$$\psi_w^0(t,y,\mathrm{X}) = \Psi_w(t,y,\mathrm{X}) + \Psi_{int}(t,x,y), \quad \mathrm{X}\in\omega_w^-. \tag{42}$$

and denote $\psi^{t,y} = \psi_w^0(t,y,\cdot)$, (14)–(15) are equivalent to the following equation: for X in $\omega^+ \cup \omega^-$,

$$Q_w(y)(\psi^{t,y},\psi^{t,y}) + Q_w(y)(\psi_m^{t,y},\psi^{t,y}) + Q_w(y)(\psi^{t,y},\psi_m^{t,y}) + \partial_X\psi^{t,y} - (\Delta_w(y))^2\psi^{t,y} = 0, \tag{43}$$

$$\begin{cases} \left[\partial_X^k \psi^{t,y}\right]_{|\sigma_w} = 0, \quad k=0,1 \\ \left[\partial_X^k \psi^{t,y}\right]_{|\sigma_w} = -\left[\partial_X^k \psi_m^{t,y}\right]_{|\sigma_w}, \quad k=2,3. \\ \psi^{t,y} \text{ 1-periodic in } Y, \quad \psi^{t,y}\big|_{X=-\gamma_w(Y)} = \dfrac{\partial\psi^{t,y}}{\partial n_w}\Big|_{X=-\gamma_w(Y)} = 0. \end{cases} \tag{44}$$

We proceed in the following way:

i) Under a smallness assumption on ϕ, we prove for all t,y the existence of a solution $\psi^{t,y}$ of (43)–(44) (in a sense to be precised).

ii) We show that $\psi^{t,y}$ belongs to $H^m(\omega^R)$, for $R>1$, and satisfies estimates (16).

iii) Under a stronger smallness assumption on ϕ, we prove that $\psi^{t,y}$ is the unique solution of (43), (44).

3.2.1 Variational formulation

In this part, we show for all (t,y) the existence of a solution $\psi^{t,y}$ of (43)-(44). More precisely, we associate a variational formulation to these equations, and show the existence of a solution for this formulation.

For all y, we define a trilinear form $b^y\left(\psi,\tilde{\psi},\varphi\right)$ on $\mathcal{D}_0^2(\omega_w)\times\mathcal{D}_0^2(\omega_w)\times\mathcal{V}_0$ by

$$b^y\left(\psi,\tilde{\psi},\phi\right) = -\int_{\omega_w}\left(\left(\nabla_w^\perp\psi\cdot\nabla_w\right)\nabla_w^\perp\tilde{\psi}\right)\cdot\nabla_w^\perp\varphi \tag{45}$$

One checks easily that b^y is well-defined for all y, satisfies

$$b^y\left(\psi,\tilde{\psi},\varphi\right) = \int Q_w(y)\left(\psi,\tilde{\psi}\right)\ \varphi,$$

for all $\psi,\tilde{\psi},\varphi$ regular enough, and that

$$b^y\left(\psi,\varphi,\varphi\right) = 0, \quad \forall\, \varphi \in \mathcal{V}_0.$$

The variational formulation associated to (43)–(44) is: for all t, y, find $\psi^{t,y} \in \mathcal{D}_0^2(\omega_w)$ *such that, for all* φ *in* $\mathcal{V}_0$,

$$b^y\left(\psi^{t,y},\psi^{t,y},\varphi\right)\ +\ b^y\left(\psi^{t,y},\psi_m^{t,y},\varphi\right)\ +\ b^y\left(\psi_m^{t,y},\psi^{t,y},\varphi\right)\ -\ \int_{\omega_w} < D_w^{2,y}\,\psi^{t,y},\ D_w^{2,y}\,\varphi >$$

$$+\int_{\omega_w}\partial_X\psi^{t,y}\,\varphi\ =\ -(1+(\chi_w'(y))^2)\left(\left[\partial_X^2\,\psi_m^{t,y}\right]_{|\sigma_w}\int_{\sigma_w}\varphi\ +\ \left[\partial_X^3\psi_m^{t,y}\right]_{|\sigma_w}\int_{\sigma_w}\partial_X\varphi\right), \quad (46)$$

where

$$D_w^{2,y}\,\psi = \begin{pmatrix}(1+(\chi_w'(y))^2)\,\partial_X^2\,\psi & -\chi_w'(y)\,\partial_{XY}^2\,\psi\\ -\chi_w'(y)\,\partial_{XY}^2\,\psi & \partial_Y^2\,\psi\end{pmatrix},$$

and $<,>$ stands for the euclidean matricial scalar product.

We now prove

Theorem 3.1 *There exists* ϕ_∞*, such that if* $\|\phi\|_\infty < \phi_\infty$*, there exists for all* $t \in [0,T]$*, for all* $y \in [y_{min}, y_{max}]$ *a function* $\psi^{t,y}$ *in* $\mathcal{D}_0^2(\omega_w)$ *satisfying* (46) *for all* φ *in* $\mathcal{V}_0$.

Proof of theorem 3.1:

The proof relies on a Galerkin scheme. Let $(\varphi_n)_n \in \mathcal{V}_0$ a basis of $\mathcal{D}_0^2(\omega_w)$. We consider the sequel of approximate problems : *For all* n*, find* ψ_n *of the form*

$$\psi_n(t,y,\mathrm{X})\ =\ \sum_{k=0}^{n}\alpha_k(t,y)\,\varphi_k(\mathrm{X}),$$

such that for all $k \in \{0,\ldots,n\}$*, for all* (t,y)*,* $\psi_n^{t,y} = \psi_n(t,y,\cdot)$ *satisfies*

$$b^y\left(\psi_n^{t,y},\psi_n^{t,y},\varphi_k\right)\ +\ b^y\left(\psi_m^{t,y},\psi_n^{t,y},\varphi_k\right)\ +\ b^y\left(\psi_n^{t,y},\psi_m^{t,y},\varphi_k\right)\ -\ \int_{\omega_w} < D_w^{2,y}\psi_n^{t,y}, D_w^{2,y}\varphi_k >$$

$$+\int_{\omega_w}\partial_X\psi_n^{t,y}\,\varphi_k = -(1+(\chi_w'(y))^2)\ \left(\left[\partial_X^2\psi_m^{t,y}\right]_{|\sigma_w}\int_{\sigma_w}\varphi_k\ +\ \left[\partial_X^3\psi_m^{t,y}\right]_{|\sigma_w}\int_{\sigma_w}\partial_X\varphi_k\right). \quad (47)$$

- A priori estimates

We start with *a priori* estimates. As t and y are simple parameters, we drop them temporarily in order to lighten notations : one can easily verify that all the constants C_i's arising in the inequalities below can be chosen independent of t and y (and n). Multiplying the last equation by α_k and summing over k leads to

$$\int_{\omega_w} < D_w^2\psi_n, D_w^2\psi_n > =\ b(\psi_n,\psi_m,\psi_n)\ +\ \int_{\omega_w}(\partial_X\psi_n)\,\psi_n$$

$$+\,(1+(\chi_w')^2)\left(\left[\partial_X^2\psi_m\right]_{|\sigma_w}\int_{\sigma_w}\psi_n + \left[\partial_X^3\psi_m\right]_{|\sigma_w}\int_{\sigma_w}\partial_X\psi_n\right). \quad (48)$$

We obtain the following energy estimates :

We have $\displaystyle\int_{\omega_w} (\partial_X \psi_n)\, \psi_n = \int_{\omega_w} \partial_X (\frac{(\psi_n)^2}{2}) = 0$

The integrals over σ_w are bounded in the following way:

$$\begin{aligned}(1+(\chi_w')^2)\Big| \big[\partial_X^2 \psi_m\big]_{|\sigma_w} \int_{\sigma_w} \psi_n + \big[\partial_X^3 \psi_m\big]_{|\sigma_w} \int_{\sigma_w} \partial_X \psi_n \Big| \\ \leq C_1\, |\phi|\, \|\psi_n\|_{H^2(\omega_w^-)} \\ \leq C_2\, |\phi|\, \|D^2 \psi_n\|_{L^2(\omega_w)},\end{aligned} \tag{49}$$

where we have used Poincaré's inequality on ω_w^-.

It remains to estimate the trilinear term, i.e. to handle terms of the type

$$I = \int_0^1 dY \int_0^{+\infty} dX\ (\partial_i \partial_j \psi_m)\ (\partial_k \psi_n)\ (\partial_l \psi_n), \quad i,j,k,l \in \{X, Y\}.$$

We write

$$\begin{aligned} I = \int_0^1 dY \int_0^{+\infty} dX\ (\partial_i \partial_j \psi_m) \left(\partial_k \psi_n(0,Y) + \left(\int_0^X dX'\ \partial_{X'} \partial_k \psi_n \right) \right) \\ \cdot \left(\partial_l \psi_n(0,Y) + \left(\int_0^X dX'\ \partial_{X'} \partial_l \psi_n \right) \right) \end{aligned}$$

so that $I = I_1 + I_2 + I_3 + I_4$, with

$$\begin{aligned} I_1 &= \int_0^1 dY\, \partial_k \psi_n(0,Y)\, \partial_l \psi_n(0,Y) \int_0^{+\infty} dX\, \partial_i \partial_j \psi_m, \\ I_2 &= \int_0^1 dY\, \partial_k \psi_n(0,Y) \int_0^{+\infty} dX\, (\partial_i \partial_j \psi_m) \left(\int_0^X dX'\, \partial_{X'} \partial_l \psi_n \right), \\ I_3 &= \int_0^1 dY\, \partial_l \psi_n(0,Y) \int_0^{+\infty} dX\, (\partial_i \partial_j \psi_m) \left(\int_0^X dX'\, \partial_{X'} \partial_k \psi_n \right), \\ I_4 &= \int_0^1 dY \int_0^{+\infty} dX\ (\partial_i \partial_j \psi_m) \left(\int_0^X dX'\, \partial_{X'} \partial_k \psi_n \right) \left(\int_0^X dX'\, \partial_{X'} \partial_l \psi_n \right). \end{aligned}$$

Through repeated use of the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |I_1| &\leq \left(\sup_Y \left(\int_0^{+\infty} dX\ |\partial_i \partial_j \psi_m| \right) \right)\ \|\psi_n\|^2_{H^{3/2}(\sigma_w)} \\ &\leq C_1\, |\phi|\, \|D^2 \psi_n\|^2_{L^2(\omega_w)}, \end{aligned}$$

$$\begin{aligned} |I_2| + |I_3| &\leq \left(\sup_Y \left(\int_0^{+\infty} dX\ |\partial_i \partial_j \psi_m| \sqrt{X} \right) \right)\ \|\psi_n\|_{H^{3/2}(\sigma_w)} \|D^2 \psi_n\|_{L^2(\omega_w)} \\ &\leq C_2\, |\phi|\, \|D^2 \psi_n\|^2_{L^2(\omega_w)}, \end{aligned}$$

$$
\begin{aligned}
|I_4| &\le \left(\sup_Y \left(\int_0^{+\infty} dX\ |\partial_i\partial_j\psi_m|X\right)\right)\ \|D^2\psi_n\|^2_{L^2(\omega_w)} \\
&\le C_3\,|\phi|\ \|D^2\psi_n\|^2_{L^2(\omega_w)},
\end{aligned}
$$

Remark: With similar computations, one has more generally that : there exists $C, \sigma > 0$, such that for all $\psi \in \mathcal{D}^2_0(\omega_w)$ and for all $R \ge 0$,

$$
\left| \int_{\omega^R} \Big(\left(\nabla_w^\perp\psi \cdot \nabla_w\right) \nabla_w^\perp\psi_m \Big) \cdot \nabla_w^\perp\psi \right| \le C\ |\phi|\ \exp(-\sigma R)\ \|D^2\psi_n\|^2_{L^2(\omega_w)} \tag{50}
$$

If we gather all these bounds, we end with the a priori estimate

$$
\begin{aligned}
\|\ D^2\psi_n\|^2_{L^2(\omega_w)} &\le\ C_5 \int_{\omega_w} < D^2_w\psi_n, D^2_w\psi_n > \\
&\le\ C_6\ |\phi|\ \Big(\|\ D^2\psi_n\|_{L^2(\omega_w)} + \|\ D^2\psi_n\|^2_{L^2(\omega_w)}\Big)
\end{aligned}
$$

Recovering the dependence on t and y, we get

$$
\|\ D^2\psi_n^{t,y}\|^2_{L^2(\omega_w)}\ \le\ \le\ C_6\ |\phi(t,y)|\ \Big(\|\ D^2\psi_n^{t,y}\|_{L^2(\omega_w)} + \|\ D^2\psi_n^{t,y}\|^2_{L^2(\omega_w)}\Big) \tag{51}
$$

In particular, if $\|\phi\|_\infty < C_6/2$, we get

$$
\|\ D^2\psi_n^{t,y}\|_{L^2(\omega_w)}\ \le\ 2\,C_6\ |\phi(t,y)|. \tag{52}
$$

- Existence of ψ_n

Let $\alpha = (\alpha_1, \ldots, \alpha_n)^t$, where $\psi_n\ =\ \sum \alpha_k\,\varphi_k$. Equations (47) read

$$
F\Big(y, \phi(t,y), \alpha(t,y)\Big) = 0, \qquad \text{with } F \text{ of type}
$$

$$
F(y,\phi,a) = Q_1(y,a,a) + Q_2(y,a,\phi) + Q_2(y,\phi,a) + A(y)\,a + L(y)\phi.
$$

The Q_i's come from the trilinear term b^y : they are smooth and bilinear in their last two variables. The matrix $A(y)$ comes from the viscous term: it is smooth in y, and invertible, as the φ_k's are independent. The matrix $L(y)$ is smooth and comes from the surface integral. Note that up to extending the functions χ_w and ϕ, one can assume that

$$
F : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \mapsto \mathbb{R}^n.
$$

Moreover, it is straightforward that for all y in $\mathbb{R}$,

$$
F(y,0,0) = 0, \quad \frac{\partial F}{\partial a}(y,0,0)\ \text{ invertible.}
$$

By the implicit function theorem, for all y, there exist neighborhoods $\mathcal{Y}_y$ of y, $\mathcal{P}_y$ of $\phi = 0$, and $\mathcal{A}_y$ of $a = 0$, and a smooth function a from $\mathcal{Y}_y \times \mathcal{P}_y$ to $\mathcal{A}_y$ such that

$$
\forall\,(\tilde y, \phi, a) \in \mathcal{Y}_y \times \mathcal{P}_y \times \mathcal{A}_y, \quad F(\tilde y,\phi,a) = 0 \quad \Leftrightarrow \quad a = a(\tilde y, \phi).
$$

As $[y_{min}, y_{max}]$ is compact,we can find a covering of $[y_{min}, y_{max}]$ by a finite number of neighborhoods $\mathcal{Y}_{y_1}, \dots, \mathcal{Y}_{y_n}$. Let

$$\mathcal{P} = \cap_{i=1}^n \mathcal{P}_{y_i}, \quad \mathcal{A} = \cap_{i=1}^n \mathcal{A}_{y_i},$$

and let $a_i : \mathcal{Y}_{y_i} \times \mathcal{P}_{y_i} \mapsto \mathcal{A}_{y_i}$ the corresponding solution of $F(y, \phi, a) = 0$, given by the implicit function theorem. Thanks to (52), there exists ϕ_∞ such that, for $\|\phi\|_\infty < \phi_\infty$, for all $t \in [0, T]$, for all $y \in [y_{min}, y_{max}]$, $\phi(t, y)$ belongs to $\mathcal{P}$ and for all i, $a_i(t, y, \phi(t, y))$ belongs to $\mathcal{A}$. It is then straightforward to see that

$$\alpha(t, y) = a_i\left(y, \phi(t, y)\right), \quad t \in [0, T], \quad y \in \mathcal{Y}_{y_i}$$

is well-defined and is a smooth solution of the approximate problem.

- Convergence of ψ_n

The estimate (52) shows that, uniformly in t and y,

$$\left(\psi_n^{t,y}\right)_n \text{ is a bounded sequence of } \mathcal{D}_0^2(\omega_w).$$

Hence, for all (t, y), there exists $\psi^{t,y} \in \mathcal{D}_0^2(\omega_w)$, such that, up to a subsequence, $\left(\psi_n^{t,y}\right)_n$ converges *weakly* to $\psi^{t,y}$, with (cf (52))

$$\|D^2 \psi^{t,y}\|_{L^2(\omega)} \;\le\; 2\, C_6\, |\phi(t, y)|.$$

Moreover, using Rellich theorem (and again up to a subsequence), $\left(\psi_n^{t,y}\right)$ converges *strongly* to $\psi^{t,y}$ in $H^s_{loc}(\omega_w)$, for any $s < 2$. It is then routine arguments to pass to the limit into (47) as n goes to infinity, to obtain: for all t, y, $\psi^{t,y}$ satisfies (46) for all $\varphi \in \mathcal{V}_0$. This ends the proof of theorem 3.1.

Remark: We do not know at this point the behaviour at infinity of solutions $\psi^{t,y}$. In particular, we still may not use $\psi^{t,y}$ as a test function in (46) (through a density argument), which would allow to conclude for uniqueness. Also, if we wish to use Sobolev injections, we need to distinguish between the oscillatory part of ψ (for which injections hold) and its average, as will be done in the next subsection.

3.2.2 H^m estimates

Let $\|\phi\|_\infty < \phi_\infty$ as in theorem 3.1, and let $\psi^{t,y} \in \mathcal{D}_0^2(\omega_w)$ satisfying (46) for all $\varphi \in \mathcal{V}_0$. Note that $\psi^{t,y}$ belongs to $H^2_{loc}(\omega_w)$. Besides, thanks to classical regularity results for elliptic operators,

$$\psi^{t,y} \in \mathcal{C}^\infty\left(\omega_w^- \cup \omega_w^+\right),$$

and satisfies (43) in the classical sense. In this part, we show that $\psi^{t,y}$ belongs to $H^m(\omega^R)$ for all $m \ge 0$ and for all $R > 1$. As t and y are simple parameters, we drop them from notations : it is straightforward to see that the various constants appearing in below inequalities can be chosen independent of t and y.

Average and oscillations

We introduce the following notations: to all $w \in L^1_{loc}\left(\mathbb{R}^+ \times \mathbb{T}\right)^N$ $(N \geq 1)$, we associate its vertical average $\overline{w} \in L^1_{loc}\left(\mathbb{R}^+\right)^N$ and its oscillatory part $w^* \in L^1_{loc}\left(\mathbb{R}^+ \times \mathbb{T}\right)^N$, through

$$\forall \mathbf{X} = (X,Y) \in \mathbb{R}^+ \times \mathbb{T}, \quad \overline{w}(X) = \int_{\Sigma(X)} w \, dY, \quad w^*(\mathbf{X}) = w(\mathbf{X}) - \overline{w}(X),$$

where for all $R \geq 0$, $\Sigma(R)$ is the cross-section at $X = R$. In order to lighten notations, we may sometimes consider $\overline{w}$ as a function of $\mathbf{X} \in \omega_w^+$ instead of $X > 0$.

The aim of this paragraph is to prove

Proposition 3.2 *There exist $C_\infty \in \mathbb{R}$ and $C, \sigma > 0$, such that for all $R > 0$, $\psi - C_\infty$ belongs to $H^2(\omega^R)$ with*

$$\|\psi - C_\infty\|_{H^2(\omega^R)} \leq C \left(\exp(-\sigma R) + \|D^2\psi^*\|_{L^2(\omega^{R/2})}\right). \tag{53}$$

Proof of proposition 3.2: As ψ belongs to $\mathcal{D}^2_0(\omega_w)$, we have

$$\psi^* \in H^2(\omega_w^+), \quad \overline{\psi}^{(2)} \in L^2(\{X > 0\}).$$

$\overline{\psi}$ satisfies for all $X > 0$ the linear differential equation

$$-\left(1 + (\chi_w')^2\right)^2 \overline{\psi}^{(4)} + \overline{\psi}^{(1)} = -\overline{Q_w(\psi,\psi)} - \overline{Q_w(\psi,\psi_m)} - \overline{Q_w(\psi_m,\psi)}.$$

Simple computations yield

$$\overline{Q_w(\psi_m,\psi)} = \overline{Q_w(\psi,\psi_m)} = 0,$$
$$\overline{Q_w(\psi,\psi)} = \overline{Q(\psi^*,\psi^*)}.$$

so that $\overline{\psi}$ satisfies

$$-\left(1 + (\chi_w')^2\right)^2 \overline{\psi}^{(4)} + \overline{\psi}^{(1)} = -\overline{Q_w(\psi^*,\psi^*)}. \tag{54}$$

Remember that

$$Q_w(\tilde{\psi},\tilde{\psi}) = \nabla_w \times \left((\nabla_w^\perp \psi \cdot \nabla_w)\nabla_w^\perp \psi\right).$$

Applying last equality to $\tilde{\psi} = \psi^*$ and integrating in the Y variable leads to

$$\overline{Q_w(\psi^*,\psi^*)} = \partial_X B(\psi^*,\psi^*)$$

for a bilinear operator B involving the derivatives of ψ^* *up to order two*. If we substitute this expression into (54) and integrate in the X variable, we finally obtain

$$-\left(1 + (\chi_w')^2\right)^2 \overline{\psi}^{(3)} + \overline{\psi} = C_\infty + B(\psi^*,\psi^*) \tag{55}$$

where C_∞ is a real constant. Any solution $\overline{\psi}$ of equation (55) is given by $\overline{\psi} = C_\infty + \varphi$, with φ a solution of

$$-\left(1 + (\chi_w')^2\right)^2 \varphi^{(3)} + \varphi = B(\psi^*,\psi^*) \tag{56}$$

This last equation can be expressed in the form of a first order system $V' = AV + F$, with

$$V = \begin{pmatrix} \varphi \\ \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (1+(\chi_w')^2)^{-2} & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ -\overline{B(\Psi^*, \Psi^*)} \end{pmatrix}. \tag{57}$$

A is diagonalisable, with eigenvalues D_w, jD_w, $j^2 D_w$, with $D_w = \left(1+(\chi_w')^2\right)^{-2/3}$. Let E_+ and E_- the eigenspaces corresponding respectively to 1 and $\{j, j^2\}$, and let P_+, P_- the associated eigenprojections. Note that $E_+ = \mathbb{C}\left(D_w^{-2}, D_w^{-1}, 1\right)^t$.

We introduce the " Green function"

$$\begin{cases} G(X) = e^{\mathbf{A}X} P^- & \text{if } X > 0, \\ G(X) = e^{-\mathbf{A}X} P^+ & \text{if } X < 0, \end{cases}$$

and

$$\begin{cases} f(X) = F(X) & \text{if } X > 0, \\ f(X) = 0 & \text{if } X < 0. \end{cases}$$

The function $G * f$ is a solution of the system $V' = AV + F$. Thus, we can write, for a $V_0 \in \mathbb{C}^4$, for all $X > 0$,

$$V(X) = e^{\mathbf{A}X} V_0 \; + \; G * f. \tag{58}$$

Note that G satisfies, for positive constants C and α

$$\forall X, \quad |G(X)| \;\leq\; C\, e^{-\alpha |X|},$$

and that $f \in L^1(\mathbb{R})$. More precisely, one has easily that for all $R > 0$,

$$\int_{\{X>R\}} |F| \;\leq\; C\, \|D^2 \psi^*\|^2_{L^2(\omega^R)}.$$

Proceeding *exactly* as in [12, page 1473], one is able to obtain

$$\|G * F\|^2_{L^2(\{X>R\})} \leq C \left(\exp(-\alpha R) \; + \; \|D^2 \Psi^*\|^2_{L^2\left(\omega^{R/2}\right)} \right) \tag{59}$$

Now, as $\overline{\psi}^{(2)} = \varphi^{(2)} \in L^2(\{X > 0\})$, we necessarily have $P_+ V_0 \in \mathbb{C}^2 \times \{0\}$. But $E_+ = \mathbb{C}\left(D_w^{-2}, D_w^{-1}, 1\right)^t$, so that $P_+ V_0 = 0$. It yields the inequality:

$$\|\varphi\|_{H^2(\{X>R\})} \;\leq\; C \; \left(\exp(-\sigma R) + \|D^2 \psi^*\|_{L^2(\omega^{R/2})} \right).$$

As $\psi - C_\infty = \varphi + \psi^*$, we finally obtain (53). This ends the proof of the proposition.

H^m estimates on $\psi - C_\infty$

In this paragraph, we prove

Proposition 3.3 *There exists $\sigma > 0$, such that for all $m \geq 0$, for all $R \geq 1$, $\psi - C_\infty \in H^m(\omega^R)$, with*

$$\|\psi - C_\infty\|_{H^m(\omega^R)} \;\leq\; C_m \left(\exp(-\sigma R) \;+\; \|\psi - C_\infty\|_{H^2(\omega^{R-1})}\right). \tag{60}$$

The keypoint of the proof is the following lemma:

Lemma 3.4 *Let $\Psi, \mathbf{f_1}, \mathbf{f_2}, \in \mathcal{C}^\infty\left(\overline{\omega_w^+}\right)$, 1-periodic in Y, satisfying*

$$(\Delta_w)^2 \Psi = \nabla_w^\perp \cdot \mathbf{f_1} \;+\; \mathbf{f_2} \qquad \textit{in } \omega_w^+,$$

where $\nabla_w = (\partial_X, \partial_Y - \chi_w' \partial_X)^t$. Then, for all $s \geq 1$, $\delta \in (0, s)$, for all $m \geq 0$, and $1 < q < \infty$,

$$\|\Psi\|_{m+3,q,s,s+1} \;\leq\; C\left(\|\mathbf{f_1}\|_{m,q,s-\delta,s+1+\delta} \;+\; \|\mathbf{f_2}\|_{m,q,s-\delta,s+1+\delta} \;+\; \|\Psi\|_{2,q,s-\delta,s+1+\delta}\right), \tag{61}$$

where $\|\cdot\|_{m,q,s,t}$ stands for the norm in $W^{m,q}\left(\omega_w^+ \cup \{s < X < t)\}\right)$, and C independent of s.

Proof of lemma 3.4 : The proof of this lemma is very close to the proof of [11, lemma 1.2], and relies on classical elliptic regularity results. Note that it is enough to prove the inequality (61) for $s = 1$: we can always come back to this case through a change of variable: $X \mapsto X - \xi$. This also implies that the constant C of the lemma is independent of s.

We content ourselves with proving:

Let ψ, f_1, $f_2 \in \mathcal{C}^\infty\left(\overline{\omega_w^+}\right)$, 1-periodic in Y, satisfying $(\Delta_w)^2\psi = \nabla_w^\perp \cdot f_1 + f_2$ in ω_w^+. Then, for all $0 < \delta < \tilde{\delta}$, for all $1 < q < \infty$,

$$\|\psi\|_{3,q,1-\delta,2+\delta} \;\leq\; C\left(\|f_1\|_{0,q,1-\tilde{\delta},2+\tilde{\delta}} \;+\; \|f_2\|_{0,q,1-\tilde{\delta},2+\tilde{\delta}} \;+\; \|\psi\|_{2,q,1-\tilde{\delta},2+\tilde{\delta}}\right). \tag{62}$$

Indeed, the proof of the proposition follows easily by induction, applying (62) for $\psi = \partial^\alpha \Psi$, $f_1 = \partial^\alpha \mathbf{f_1}$, $f_2 = \partial^\alpha \mathbf{f_2}$, $|\alpha| = m$, and appropriate $\delta, \tilde{\delta}$.

Let $0 < \delta < \tilde{\delta}$ and $1 < q < \infty$ fixed. Let $0 \leq \zeta \leq 1$ be a $\mathcal{C}^\infty$ function on $\mathbb{R}^3$ such that

$$\begin{aligned} &\zeta = 1 \quad \text{in } K = [1-\delta, 2+\delta] \times \left[-\frac{1}{2}, \frac{3}{2}\right] \\ &\zeta = 0 \quad \text{outside a } \mathcal{C}^\infty \text{ open set } O \text{ with } K \subset O \Subset \; s]1-\tilde{\delta}, 2+\tilde{\delta} \times] -1, 2[. \end{aligned}$$

We set $\tilde{\psi} = \zeta\psi$, $\tilde{u} = \nabla_w^\perp \tilde{\psi}$, and $u = \nabla_w^\perp \psi$. We derive the following system on $\tilde{u}$:

$$\begin{aligned} \nabla_w^\perp \cdot (\Delta_w \tilde{u}) = \nabla_w^\perp \cdot \tilde{f}_1 + f_2 \quad &\text{in } O, \\ \nabla_w \cdot \tilde{u} = 0 \quad &\text{in } O, \\ \tilde{u} = 0 \quad &\text{on } \partial O, \end{aligned} \tag{63}$$

where

$$\tilde{f}_1 = (4\nabla_w \zeta \cdot \nabla_w)\, u \;+\; \zeta f_1,$$

$$\begin{aligned} \tilde{f}_2 \;=\; &\zeta f_2 \;+\; \Delta_w\left((\Delta_w \zeta)\psi\right) \;+\; 2\left(\nabla_w \Delta_w \zeta \cdot \nabla_w\right)\psi \\ &+\; \left(\nabla_w(\nabla_w \zeta) \cdot \nabla_w(\nabla_w \psi)\right) \;-\; 4\nabla_w(\nabla_w\zeta) \cdot \nabla_w u \;-\; \nabla_w^\perp \cdot f_1. \end{aligned} \tag{64}$$

Using [11, theorem 3.2, p130], there exists $G \in W_0^{1,q}(O)$ such that

$$\nabla \cdot G = \tilde{f}_2 \quad \text{in O, with} \quad \|G\|_{W^{1,q}(O)} \le C\|\tilde{g}\|_{L^q(O)}.$$

The function $F = (G_2, -G_1 + \chi_w' G_2)$, where $G = (G_1, G_2)^t$ satisfies

$$\nabla_w \times F = \tilde{f}_2, \qquad \text{with} \ \|F\|_{W^{1,q}(O)} \le \|\tilde{f}_2\|_{L^q(O)}.$$

Going from the vorticity formulation (63) to a velocity formulation, we obtain

$$\begin{aligned} \Delta_w \tilde{u} \ + \ \nabla_w q \ = \ \tilde{f}_1 + F \quad \text{in } O, \\ \nabla_w \cdot \tilde{u} = 0 \quad \text{in } O, \\ \tilde{u} = 0 \quad \text{on } \partial O. \end{aligned} \tag{65}$$

Classical estimates for elliptic systems yield

$$\|\tilde{u}\|_{W^{2,q}(O)} \le C\left(\|\tilde{f}_1 + F\|_{L^q(0)} + \|\tilde{u}\|_{L^q(O)}\right),$$

which leads easily to (62).

Proof of proposition 3.3: To prove this proposition, we work with the function $\psi_\infty := \psi - C_\infty - \psi_m$, which satisfies on ω_w^+ the equation

$$Q_w(\psi_\infty, \psi_\infty) \ + \ \partial_X \psi_\infty \ - \ \Delta_w^2 \psi_\infty = 0,$$

which can also be written, introducing $u_\infty = \nabla^\perp \psi_\infty$,

$$\Delta_w^2 \psi_\infty = \nabla_w^\perp \times (-u_\infty \cdot \nabla_w u_\infty) \ - \ \partial_X \psi_\infty. \tag{66}$$

Let us show that: *for all $\delta > 0$, there exists $C > 0$ such that*

$$\forall\, R \ge 1, \quad \|\psi_\infty\|_{H^3(\omega^R)} \le C\|\psi_\infty\|_{H^2(\omega^{R-\delta})}. \tag{67}$$

Let $\delta > 0$, For all $s \ge 1$, we have

$$\begin{aligned} \|-u_\infty \cdot \nabla_w u_\infty\|_{0,3/2,s-\delta,s+1+\delta}\| &\le C_1\|u_\infty\|_{0,6,s-\delta,s+1+\delta}\,\|\nabla_w u_\infty\|_{0,2,s-\delta,s+1+\delta} \\ &\le C_2\|\psi_\infty\|^2_{2,2,s-\delta,s+1+\delta} \le C_3\|\psi_\infty\|_{2,2,s-\delta,s+1+\delta} \end{aligned} \tag{68}$$

Note that the constants C_i do not depend on s (thanks to the invariance by translation along X). We also have

$$\|\partial_X \psi_\infty\|_{0,3/2,s-\delta,s+1+\delta} \le C_4\|\partial_X \psi_\infty\|_{0,2,s-\delta,s+1+\delta} \le C_5\|\psi_\infty\|_{2,2,s-\delta,s+1+\delta},$$

so that applying lemma 3.4, we have

$$\begin{aligned} \|\psi_\infty\|_{3,3/2,s-\delta/2,s+1+\delta/2} &\le C_6\|u_\infty \cdot \nabla_w u_\infty + \partial_X \psi_\infty\|_{0,3/2,s-\delta,s+1+\delta} \\ &\le C_7\|\psi_\infty\|_{2,2,s-\delta,s+1+\delta}. \end{aligned} \tag{69}$$

We now iterate the process and get an improved regularity. By Sobolev imbedding, we have

$$\|u_\infty\|_{0,\infty,s-\delta/2,s+1+\delta/2} \le C_8\|\psi_\infty\|_{3,3/2,s-\delta/2,s+1+\delta/2} \le C_9\|\psi_\infty\|_{2,2,s-\delta,s+1+\delta}.$$

Hence, we get the following bound

$$\begin{aligned} \| -u_\infty \cdot \nabla_w u_\infty \|_{0,2,s-\delta/2,s+1+\delta/2} \| &\le C_{10} \|u_\infty\|_{0,\infty,s-\delta/2,s+1+\delta/2} \, \|\nabla_w u_\infty\|_{0,2,s-\delta/2,s+1+\delta/2} \\ &\le C_{11} \|\psi_\infty\|^2_{2,2,s-\delta,s+1+\delta} \le C_{12} \|\psi_\infty\|_{2,2,s-\delta,s+1+\delta}. \end{aligned} \tag{70}$$

Moreover,

$$\|\partial_X \psi_\infty\|_{0,2,s-\delta/2,s+1+\delta/2} \le \|\psi_\infty\|_{2,2,s-\delta,s+1+\delta},$$

so that applying again lemma 3.4,

$$\begin{aligned} \|\psi_\infty\|_{3,2,s,s+1} &\le C \|u_\infty \cdot \nabla_w u_\infty + \partial_X \psi_\infty\|_{0,2,s-\delta/2,s+1+\delta/2} \\ &\le C' \|\psi_\infty\|_{2,2,s-\delta,s+1+\delta}, \end{aligned} \tag{71}$$

with C' independent of s. Using this last inequality with $s = R + k$, $k = 1, 2, \ldots$, and summing over k, we get (67).

Using the same type of bootstrap arguments with the derivatives of ψ_∞ instead of ψ_∞, it is easy to show that more generally: for all $m \ge 3$, for all $\delta > 0$, there exists $C > 0$ such that:

$$\forall\, R \ge 1, \quad \|\psi_\infty\|_{H^m(\omega^R)} \le C \|\psi_\infty\|_{H^2(\omega^{R-\delta})}. \tag{72}$$

Back to $\psi - C_\infty = \psi_\infty + \psi_m$, the proposition follows. It ends the proof.

Control of ψ at infinity

To show that ψ is in $H^m(\omega^R)$ for all $R > 0$, it remains to prove that the constant C_∞ of proposition 3.2 is zero.

On one hand, we multiply equation (43) by $\psi(= \psi^{t,y})$ and integrate on $\omega \cap \{X < R\}$. After a few integration by parts, using jump conditions (44), we are left with

$$\begin{aligned} \int_{\omega \cap \{X<R\}} \langle D^2_w \psi, D^2_w \psi \rangle &= - \int_{\omega \cap \{X<R\}} \Big(\left(\nabla_w^\perp \psi \cdot \nabla_w\right) \nabla_w^\perp \psi \Big) \cdot \nabla_w^\perp \psi \\ &- \int_{\omega \cap \{X<R\}} \Big(\left(\nabla_w^\perp \psi_m \cdot \nabla_w\right) \nabla_w^\perp \psi \Big) \cdot \nabla_w^\perp \psi \; - \int_{\omega \cap \{X<R\}} \Big(\left(\nabla_w^\perp \psi_m \cdot \nabla_w\right) \nabla_w^\perp \psi \Big) \cdot \nabla_w^\perp \psi_m \\ &+ \frac{\psi(R,Y)^2}{2} + \int_{\sigma(R)} \partial_X(\Delta_w \psi)\, \psi \; - \int_{\sigma(R)} (\nabla_w \partial_X \psi) \cdot \nabla_w \psi \\ &+ \int_{\sigma(R)} \begin{pmatrix} \chi_w' \\ 1 \end{pmatrix} \cdot \Big(\left(\nabla_w^\perp (\psi + \psi_m) \cdot \nabla_w\right) \nabla_w^\perp \psi \Big) + \int_{\sigma(R)} \begin{pmatrix} \chi_w' \\ 1 \end{pmatrix} \cdot \Big(\left(\nabla_w^\perp \psi \cdot \nabla_w\right) \nabla_w^\perp \psi_m \Big) \\ &+ (1 + (\chi_w')^2) \left(\Big[\partial_X^2 \psi_m\Big]_{|\sigma_w} \int_{\sigma_w} \psi + \Big[\partial_X^3 \psi_m\Big]_{|\sigma_w} \int_{\sigma_w} \partial_X \psi \right). \end{aligned} \tag{73}$$

As $\psi - C_\infty$ belongs to $H^m(\omega^R)$ for all $R > 1$, we may pass easily to the limit as R goes to infinity in (73) and obtain

$$\begin{aligned} \int_{\omega_w} \langle D^2_w \psi, D^2_w \psi \rangle \; &= \; b(\psi, \psi_m, \psi) \; + \; \frac{C_\infty^2}{2} \\ &+ \; (1 + (\chi_w')^2) \left(\Big[\partial_X^2 \psi_m\Big]_{|\sigma_w} \int_{\sigma_w} \psi + \Big[\partial_X^3 \psi_m\Big]_{|\sigma_w} \int_{\sigma_w} \partial_X \psi \right). \end{aligned} \tag{74}$$

On the other hand, we use the fact that, up to a subsequence

$$\psi_n \to \psi, \quad \text{weakly in } \mathcal{D}_0^2(\omega_w),$$

where ψ_n was built in section 3.2.1. Hence,

$$\int_{\omega_w} \langle D_w^2\psi, D_w^2\psi\rangle \leq \liminf_{n\to+\infty} \int_{\omega_w} \langle D_w^2\psi_n, D_w^2\psi_n\rangle.$$

Using (48) yields

$$\liminf_{n\to+\infty} \int_{\omega_w} \langle D_w^2\psi_n, D_w^2\psi_n\rangle = \liminf_{n\to+\infty} b(\psi_n,\psi_m,\psi_n)$$
$$+ \liminf_{n\to+\infty} (1+(\chi_w')^2)\left(\left[\partial_X^2\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \psi_n + \left[\partial_X^3\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \partial_X\psi_n\right). \quad (75)$$

By the Rellich theorem, for all $s>2$, we have up to a subsequence

$$\psi_n \to \psi, \quad \text{strongly in } H^s_{loc}. \quad (76)$$

If we choose for instance $s = 7/4$, and thanks to the imbedding

$$H^s(\omega_w^-) \hookrightarrow H^1(\sigma_w),$$

we deduce

$$\liminf_{n\to+\infty} (1+(\chi_w')^2)\left(\left[\partial_X^2\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \psi_n + \left[\partial_X^3\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \partial_X\psi_n\right) =$$
$$(1+(\chi_w')^2)\left(\left[\partial_X^2\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \psi + \left[\partial_X^3\psi_m\right]_{|\sigma_w} \int_{\sigma_w} \partial_X\psi\right) \quad (77)$$

It remains to show that

$$\liminf_{n\to+\infty} b(\psi_n,\psi_m,\psi_n) = b(\psi,\psi_m,\psi).$$

Thanks to (50),

$$\left|\int_{\omega^R} \left(\left(\nabla_w^\perp\psi_n\cdot\nabla_w\right)\nabla_w^\perp\psi_m\right)\cdot\nabla_w^\perp\psi_n\right| \leq C_1\,|\phi|\,\exp(-\sigma R)\,\|D^2\psi_n\|^2_{L^2(\omega_w)}$$
$$\leq C_2'\,\exp(-\sigma R), \quad (78)$$

where we have used the uniform bound (52). In the same way

$$\left|\int_{\omega^R} \left(\left(\nabla_w^\perp\psi\cdot\nabla_w\right)\nabla_w^\perp\psi_m\right)\cdot\nabla_w^\perp\psi\right| \leq C_3\,\exp(-\sigma R).$$

Thus, for all $\varepsilon>0$, there exists R such that

$$|b(\psi_n,\psi_m,\psi_n) - b(\psi,\psi_m,\psi)| \leq \varepsilon + \left|\int_{\omega\cap\{X<R\}} \left(\left(\nabla_w^\perp\psi_n\cdot\nabla_w\right)\nabla_w^\perp\psi_n\right)\cdot\nabla_w^\perp\psi_m - \right.$$
$$\left.\int_{\omega\cap\{X<R\}} \left(\left(\nabla_w^\perp\psi\cdot\nabla_w\right)\nabla_w^\perp\psi\right)\cdot\nabla_w^\perp\psi_m\right| \quad (79)$$

The second term at the right handside goes to zero as $n \to +\infty$, using (76). Finally,

$$\int_{\omega_w} \langle D^2_w \psi, D^2_w \psi \rangle \;\le\; b(\psi, \psi_m, \psi) + (1 + (\chi'_w)^2) \left(\left[\partial^2_X \psi_m \right]_{|\sigma_w} \int_{\sigma_w} \psi + \left[\partial^3_X \psi_m \right]_{|\sigma_w} \int_{\sigma_w} \partial_X \psi \right). \quad (80)$$

The comparison with (74) yields $C_\infty = 0$.

Exponential decay

In this last paragraph, we prove the so-called "Saint-Venant" estimates on ψ, that means exponential decay estimates (16). Once again, we rather work with $\Psi = \psi - \psi_m$, which satisfies on ω_w^+ the equation

$$Q_w(\Psi, \Psi) \;+\; \partial_X \Psi \;-\; \Delta^2_w \Psi = 0,$$

Combining propositions 3.2 and 3.3, it is enough to prove

Proposition 3.5 *There exists R_1, C, $\sigma > 0$, such that*

$$\forall R \ge R_1, \quad \|D^2 \Psi^*\|_{L^2(\omega^R)} \le C \exp(-\sigma R).$$

Proof of proposition 3.5: Let $f(R) = \displaystyle\int_{\omega^R} |D^2 \psi^*|^2$. An energy estimate on the previous equation gives

$$f(R) \le C \int_{\omega^R} \langle D^2_w \Psi^*, D^2_w \Psi^* \rangle = C \sum_{k=1}^{6} I_k, \quad (81)$$

with

$$I_1 = \int_{\omega^R} \partial_X \Psi^* \; \Psi^*, \quad I_2 = \int_{\omega^R} Q_w(\overline{\Psi}, \Psi^*) \; \Psi^*, \quad I_3 = \int_{\omega^R} Q_w(\Psi^*, \overline{\Psi}) \; \Psi^*,$$

$$I_4 = \int_{\omega^R} Q_w(\Psi^*, \Psi^*) \; \Psi^*, \quad I_5 = -\int_{\sigma(R)} ((\chi'_w)^2 + 1)(\partial_X \Delta_w \Psi^*) \; \Psi^*,$$
$$I_6 = +\int_{\sigma(R)} (1 + (\chi'_w)^2) \partial_X \left(|\nabla_w \Psi^*|^2 \right). \quad (82)$$

$$I_1 = -\int_{\sigma(R)} \frac{|\Psi^*|^2}{2} \le 0.$$

We have

$$\begin{aligned} I_4 &= \int_R^{+\infty} \int_{\sigma(R')} \left(\nabla^\perp_w \Psi^* \cdot \nabla_w \Delta_w \Psi^* \right) \; \Psi^* \\ &\le \left(\sup_{Y, X \ge R} |\nabla_w \Delta_w \Psi^*| \right) \int_R^{+\infty} \left(\int_{\sigma(R')} |\Psi^*|^2 \right)^{1/2} \left(\int_{\sigma(R')} |\nabla^\perp_w \Psi^*|^2 \right)^{1/2} \end{aligned} \quad (83)$$

Using proposition 3.3, we obtain for R large enough $\sup_{Y,Z\geq R} |\nabla_w \Delta_w \Psi^*| \leq \delta$, where δ positive will be chosen later. Moreover,

$$\int_{\sigma(R')} |\Psi^*|^2 + \int_{\sigma(R')} |\nabla_w^\perp \Psi^*|^2 \leq 2 \int_{\sigma(R')} |D^2 \Psi^*|^2.$$

Finally,

$$I_4 \leq \delta f(R).$$

The estimates of I_2 and I_3 are similar to the one of I_4. We get for R large enough

$$I_2 + I_3 \leq \delta f(R).$$

We treat I_5 in the following way:

$$\begin{aligned} I_5 &= -\int_{\sigma(R)} (1+(\chi_w')^2)\,(\partial_X \Delta_w \Psi^*)\;\Psi^* \\ &= -(1+(\chi_w')^2)\,\partial_X \left(\int_{\sigma(R)} \Delta_w \Psi^*\;\Psi^* \right) + (1+(\chi_w')^2) \int_{\sigma(R)} \Delta_w \Psi^*\,\partial_X \Psi^* \\ &\leq -(1+(\chi_w')^2)\,\partial_X \left(\int_{\sigma(R)} \Delta_w \Psi^*\;\Psi^* \right) + C \int_{\sigma(R)} |D^2 \Psi^*|^2. \\ &\leq -(1+(\chi_w')^2)\,\partial_X \left(\int_{\sigma(R)} \Delta_w \Psi^*\;\Psi^* \right) - C f'(R) \end{aligned} \tag{84}$$

Finally,

$$I_6 = \left(1+(\chi_w')^2\right)\,\partial_X \int_{\sigma(R)} |\nabla_w \Psi^*|^2.$$

Back to the main energy estimate (81), we obtain for R large enough.

$$\begin{aligned} (1-2\delta) f(R) \leq -C f'(R) - (1+(\chi_w')^2)\,\partial_X \left(\int_{\sigma(R)} \Delta_w \Psi^*\;\Psi^* \right) \\ + \left(1+(\chi_w')^2\right)\,\partial_X \left(\int_{\sigma(R)} |\nabla_w \Psi^*|^2 \right), \end{aligned} \tag{85}$$

where C is a positive constant. We integrate this inequality between $t > 0$ large enough and $+\infty$. It leads to

$$\int_t^{+\infty} f(R)\,dR + C_1 f'(t) \leq C_2' f(t), \tag{86}$$

where the C_i's are positive constants. We conclude thanks to [11, lemma 2.2].

3.2.3 Uniqueness

In this part, we show for all (t,y) the uniqueness of the solution $\psi^{t,y} \in \mathcal{H}^2_0(\omega_w)$ of (43)–(44), under a smallness assumption on ϕ. Let (t,y) fixed, and $\psi_1^{t,y}$, $\psi_2^{t,y}$ in $\mathcal{H}^2_0(\omega_w)$ two solutions of (43)–(44), and $\theta = \psi_1^{t,y} - \psi_2^{t,y}$. It satisfies, for all $\varphi \in \mathcal{V}_0$,

$$-\int_{\omega_w} < D^2_w\theta, D^2_w\theta > \; + b^y(\theta, \psi_2^{t,y}, \varphi) \; + \; b^y(\psi_1^{t,y}, \theta, \varphi) \; + \; b^y(\theta, \psi_m^{t,y}, \varphi) \; + \; b^y(\psi_m^{t,y}, \theta, \varphi)$$
$$+ \int_{\omega_w} (\partial_X \theta)\, \varphi = 0 \quad (87)$$

As $\theta \in \mathcal{H}^2_0(\omega_w)$, there exists φ_n in $\mathcal{V}_0$ such that $\varphi_n \to \theta$ in $\mathcal{H}^2_0(\omega_w)$. It is then easy to pass to the limit in above equation, as n goes to infinity, with $\varphi = \varphi_n$. It yields

$$-\int_{\omega_w} < D^2_w\theta, D^2_w\theta > \; + b(\theta, \psi_m, \theta) = 0.$$

Using (50), we obtain,

$$|b(\theta, \psi_m, \theta)| \; \le \; C\, |\phi| \, \|D^2\theta\|^2_{L^2(\omega_w)},$$

so that

$$\|D^2\theta\|^2_{L^2(\omega_w)} \; \le \; C'|\phi(t,y)| \, \|D^2\theta\|^2_{L^2(\omega_w)}.$$

Finally, for $\|\phi\|_\infty \le \phi_\infty < 1/C'$, we obtain $\theta = 0$, which concludes the proof of the uniqueness.

We thus may define, for $\|\phi\|_\infty < \phi_\infty$ a function $\psi^0_w = \psi^0_w(t,y,\mathrm{X})$ by:

for all (t,y), $\psi^0_w(t,y,\cdot)$ is the only solution in $\mathcal{H}^2_0(\omega_w)$ of (43)–(44).

As already mentioned, if we come back to the original system (14), (15) and define $\Psi_w = \Psi_w(t,y,\mathrm{X})$ by (41)-(42), it is clear that the function Ψ_w satisifies the statements of theorem 2.1. This ends the proof of theorem 2.1.

We end section 3.2 by additional regularity results on ψ^0_w with respect to t and y. Indeed, to build the other profiles of expansion (21) requires the existence and control of $\partial_t^\alpha \partial_y^\beta \Psi^0_w$ (or equivalently $\partial_t^\alpha \partial_y^\beta \psi^0_w$). This is possible thanks to

Proposition 3.6 *For all $k \ge 0$, there exists $\phi^k_\infty > 0$, such that if $\|\phi\|_\infty < \phi^k_\infty$, then*

i) $\psi^0_w \in W^{k,\infty}\left((0,T) \times (y_{min}, y_{max}); \mathcal{D}^2_0(\omega_w)\right)$.

ii) *For all α, β, $|\alpha| + |\beta| \le k$, for all $R > 1$, $\partial_t^\alpha \partial_y^\beta \psi^0_w(t,y,\cdot)$ belongs to $H^m(\omega^R)$, with*

$$\sup_{t,y} \|\partial_t^\alpha \partial_y^\beta \psi^0_w(t,y,\cdot)\|_{H^m(\omega^R)} \; \le \; C_m \exp(-\sigma R), \quad R \ge R_1.$$

Proof of proposition 3.6: We only give the main ideas of the proof. Indeed, precise computations are very close to those performed above, so that we do not give all the details

- Case $k = 0$

Let $\psi_n = \psi_n(t,y,\mathrm{X})$ the approximation built in subsection 3.2. On one hand, estimate (52) shows that

$$(\psi_n)_n \text{ is a bounded sequence of } E := L^\infty\left((0,T)\times(y_{min},y_{max});\, \mathcal{D}_0^2(\omega_w)\right).$$

Thus, there exists a subsequence, that we still denote $(\psi_n)_n$ which converges for the weak star topology to a function ψ^0 of E. In particular, for any compact subset K of ω_w, we get

$$\forall\, \varphi \in \mathcal{C}^\infty([0,T]\times[y_{min},y_{max}]\times K), \qquad \int \psi_n\varphi \xrightarrow[n\to+\infty]{} \int \psi^0\varphi.$$

On the other hand, as $\psi_w^0(t,y,\cdot)$ is the only solution of (43)–(44), one shows through standard arguments that for all (t,y), for all $s<2$,

$$\psi_n(t,y,\cdot)\to\psi_w^0(t,y), \quad n\to+\infty, \qquad \text{strongly in } H^s(K).$$

Thanks to this last convergence, we also get

$$\forall\, \varphi \in \mathcal{C}^\infty([0,T]\times[y_{min},y_{max}]\times K), \qquad \int \psi_n\varphi \xrightarrow[n\to+\infty]{} \int \psi_w^0\varphi.$$

Finally, $\psi_w^0 \equiv \psi_0$, and therefore belongs to E.

- Case $k=1$

Let (h_n) a sequence going to zero. The idea is to consider the sequence $(\varphi_n = \varphi_n(t,y,\mathrm{X}))_n$ given by

$$\varphi_n(t,y,\cdot) = \frac{\psi_w^0(t+h_n,y,\cdot)-\psi_w^0(t,y,\cdot)}{h_n}.$$

Note that φ_n converges to $\partial_t\psi_w^0$ in $\mathcal{D}'((0,T)\times(y_{min},y_{max})\times\omega_w)$. Moreover, repeating the whole process above with φ_n instead of ψ_n, one shows that, for $\|\phi\|_\infty$ small enough, this sequence converges as above, to a function $p=p(t,y,\mathrm{X})\in E$. For all t,y, $p(t,y,\cdot)$ is the unique solution of equations that are of the same type as (43)–(44). In particular, it satisfies exponential decay estimates of type (16).

Finally, one has $\partial_t\psi_w^0 = p \in E$, with appropriate exponential decay estimates. In the same way, one has also $\partial_y\psi_w^0\in E$, with appropriate exponential decay estimates.

- Case $k>1$

One applies inductively the same arguments, replacing ψ_w^0 by its derivatives.

3.3 Convergence result

This part is devoted to the proof of theorem 2.2. The scheme of the proof is classical: on the basis of the previous analysis, we build an approximate solution Ψ^ε_{app} of the type (21) with $n=1$. We then perform energy estimates on $v^\varepsilon = \nabla^\perp(\Psi^\varepsilon - \Psi^\varepsilon_{app})$.

3.3.1 The approximate solution

Interior and eastern profiles

We remind that Ψ^0 is given by (29), and that $\Psi^0_e \equiv 0$. The next profiles are built inductively. Assume that the Ψ^j and Ψ^j_e, $j \le i-1$, are well-defined, with Ψ^k_e and its derivatives decaying exponentially to zero. Then, the i^{th} interior profile satisfies

$$\partial_X \Psi^i = F^i,$$

where F^i depends on the Ψ^j, $j \le i-1$. This defines Ψ^i, up to the addition of a function of t and y. To determine this function we use the equation on the eastern profile Ψ^i_e. This profile satisfies a *linear elliptic* equation of the type

$$-\Delta_e^2 \Psi^i_e \;-\; \partial_X \Psi^i_e \;=\; G^i, \quad \mathrm{X} \in \omega_e^+ \cup \omega_e^-, \tag{88}$$

fulfilled with jump and boundary conditions

$$\left[\Psi^i_e(t,y,\cdot)\right]_{|\sigma_e} = -\left[\Psi^i(t,y,\cdot)\right]_{|\{x=\chi_e(y)\}}, \tag{89}$$

$$\begin{aligned} &\left[\partial_X^k \Psi^i_e(t,y,\cdot)\right]_{|\sigma_e} = g^{i,k}(t,y), \quad k=1,\ldots,3 \\ &\Psi^i_e(t,y,\cdot) \;\text{ 1-periodic in }\; Y, \\ &\Psi^i_e(t,y,\cdot)|_{X=\gamma_e(Y)} = 0, \quad \partial_{n_w} \Psi^i_e(t,y,\cdot)|_{X=\gamma_e(Y)} = h^i, \end{aligned} \tag{90}$$

where the G^i, $g^{i,k}$, and h^i depend on the Ψ^j and Ψ^j_e, $j \le i-1$. If we introduce

$$\psi^i_e = \Psi^i_e + \left[\Psi^i\right]_{|\{x=\chi_e(y)\}}, \quad \mathrm{X} \in \omega_e^+,$$
$$\psi^i_e = \Psi^i_e, \quad \mathrm{X} \in \omega_e^-,$$

ψ^i_e is formally solution of (88), (90), with the homogeneous jump condition

$$\left[\Psi^i_e(t,y,\cdot)\right]_{|\sigma_e} = 0 \tag{91}$$

Using Lax-Milgram lemma, it is then direct to show the existence and uniqueness of ψ^i_e solution of (88), (90), (91). Moreover, using the same kind of arguments than in subsection 3.3.2., one shows that there exists $C = C(t,y)$ such that $\psi^i_e(t,y,\cdot) + C(t,y)$ and its derivatives decay exponentially to zero. Hence, we choose Ψ^i such that

$$\left[\Psi^i(t,y,\cdot)\right]_{|\{x=\chi_e(y)\}} = C(t,y),$$

so that $\Psi^i_e(t,y,\cdot)$ decays exponentially to zero. This completes the construction of Ψ^i and Ψ^i_e.

Western profiles

The first western profile Ψ^0_w has been built in section 3.2. The second profile Ψ^1_w can be obtained by solving similar equations, taking into account the error term created by Ψ^0_w. Such error term is well controlled, thanks to proposition 3.6. More generally, the following profiles are solutions of similar equations, with error terms coming from lower profiles. As the construction is straightforward but tedious, we will not detail it.

3.3.2 Final energy estimates

We now focus on the energy estimates on the difference between the exact and the approximate solutions. Let Ψ^ε_{app} the approximation above, which is well defined (and regular enough) for $\|\phi\|_\infty$ small enough. Let $u^\varepsilon_{app} = \nabla^\perp \Psi^\varepsilon_{app}$, and $u^\varepsilon = \nabla^\perp \Psi^\varepsilon$, with Ψ^ε satisfying (3). We wish to show that

$$\|\Psi^\varepsilon - \Psi^\varepsilon_{app}\|_{L^\infty_t(H^1(\Omega^\varepsilon)))} \xrightarrow[\varepsilon\to 0]{} 0,$$

or equivalently that

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^\infty_t(L^2(\Omega^\varepsilon)))} \xrightarrow[\varepsilon\to 0]{} 0.$$

The difference $v^\varepsilon = u^\varepsilon - u^\varepsilon_{app}$ belongs to $H^1(\Omega^\varepsilon)$, and satisfies

$$\begin{aligned}
\partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon\cdot\nabla u^\varepsilon_{app} + u^\varepsilon_{app}\cdot\nabla v^\varepsilon \qquad & & \\
+v^\varepsilon + (\eta_B+\beta y)(v^\varepsilon)^\perp + \nabla q^\varepsilon - \Delta v^\varepsilon &= R^\varepsilon && \text{in } \Omega\cup\Omega^\varepsilon_w\cup\Omega^\varepsilon_e,\\
\nabla\cdot v^\varepsilon &= 0 && \text{in } \Omega\cup\Omega^\varepsilon_w\cup\Omega^\varepsilon_e,\\
[v^\varepsilon]_{|\Sigma_w\cup\Sigma_e} &= 0 && \text{at } \Sigma_w\cup\Sigma_e,\\
\left[\frac{\partial v^\varepsilon}{\partial n} - q^\varepsilon\vec{n}\right]_{|\Sigma_w\cup\Sigma_e} &= \sigma^\varepsilon && \text{at } \Sigma_w\cup\Sigma_e,\\
v^\varepsilon &= \varphi^\varepsilon && \text{at } \Gamma^\varepsilon_w\cup\Gamma^\varepsilon_e
\end{aligned} \tag{92}$$

with

$$\begin{aligned}
\|R^\varepsilon\|_{L^\infty_t(H^{-1}(\Omega^\varepsilon))} = O(\varepsilon), &\quad \|\sigma^\varepsilon\|_{L^\infty_t\left(H^{1/2}(\Sigma_w\cup\Sigma_e)\right)} = O(\varepsilon),\\
\|\varphi^\varepsilon\|_{W^{1,\infty}_t\left(H^{1/2}(\Gamma^\varepsilon_w\cup\Gamma^\varepsilon_e)\right)} = O\left(\exp\left(-\frac{\lambda}{\varepsilon}\right)\right), &\quad \lambda>0.
\end{aligned}$$

The boundary term φ^ε is the trace of the western (resp. eastern) boundary layer at Γ^ε_e (resp. at Γ^ε_w):

$$\begin{aligned}
\varphi^\varepsilon|_{\Gamma^\varepsilon_e} &= \varphi^\varepsilon_w(t,y,\chi_e(y)+\varepsilon\gamma_e(\varepsilon^{-1}y))|_{\Gamma^\varepsilon_e}\\
\varphi^\varepsilon|_{\Gamma^\varepsilon_w} &= \varphi^\varepsilon_e(t,y,\chi_w(y)-\varepsilon\gamma_w(\varepsilon^{-1}y))|_{\Gamma^\varepsilon_w}
\end{aligned}$$

with φ^ε_w, φ^ε_e, and their derivatives exponentially decaying in their last variable. This explains the exponential bound on φ^ε. Note that thanks to (11), φ^ε_e and φ^ε_w are zero for $y\in[y_{max}-\lambda, y_{max}]\cup[y_{min}, y_{min}+\lambda]$.

Lift of the boundary term

We must add a corrector to v^ε so as to correct the boundary conditions. Let $\Gamma^\varepsilon_w(y) = \chi_w(y) - \varepsilon\gamma_w(\varepsilon^{-1}y)$, $\quad\Gamma^\varepsilon_e(y) = \chi_e(y)+\varepsilon\gamma_e(\varepsilon^{-1}y)$. We set

$$\begin{aligned}
w^\varepsilon_1(t,x,y) \;=\; & \left(\varphi^\varepsilon_e(t,y,\Gamma^\varepsilon_w(y))\,\frac{(2x-\Gamma^\varepsilon_w(y))^2+1}{(\Gamma^\varepsilon_w(y))^2+1}\,\theta(x-\Gamma^\varepsilon_w(y))\right)\\
& + \left(\varphi^\varepsilon_w(t,y,\Gamma^\varepsilon_e(y))\,\frac{(2x-\Gamma^\varepsilon_e(y))^2+1}{\Gamma^\varepsilon_e(y)^2+1}\,\theta(x-\Gamma^\varepsilon_e(y))\right)
\end{aligned}$$

where θ belongs to $\mathcal{C}_c^\infty([-\delta,\delta])$ for $\delta>0$ small enough.

It is clear that

$$\|w_1^\varepsilon\|_{W_t^{1,\infty}(H^1(\Omega^\varepsilon))} \;\le\; C\,\exp(-\lambda'/\varepsilon), \quad \lambda'>0.$$

Now, we may apply [11, Lemma 3.1]: there exists $w_2^\varepsilon \in L^\infty\left(H_0^1(\Omega^\varepsilon)\right)^3$ such that

$$\nabla\cdot w_2^\varepsilon = -\nabla\cdot w_1^\varepsilon, \quad \|w_2^\varepsilon\|_{L_t^\infty(H^1(\Omega^\varepsilon))} \;\le\; C(\Omega^\varepsilon)\,\|\nabla\cdot w_1^\varepsilon\|_{L_t^\infty(L^2(\Omega^\varepsilon))}.$$

Moreover, we can choose

$$C(\Omega^\varepsilon) = C\,\delta(\Omega^\varepsilon)^2\,\left(1+\delta(\Omega^\varepsilon)\right) \le C'$$

where δ is the Lebesgue measure on $\mathbb{R}^2$, and C' is independent of ε. A look at the proof also shows that the same inequality holds for time derivatives, so that finally

$$\|w_2^\varepsilon\|_{W_t^{1,\infty}(H^1(\Omega^\varepsilon))} \le\; C'\varepsilon^2.$$

If we set $v^\varepsilon = w^\varepsilon + w_1^\varepsilon + w_2^\varepsilon$, w^ε is solution of

$$\begin{aligned}
\partial_t w^\varepsilon + w^\varepsilon\cdot\nabla w^\varepsilon + w^\varepsilon\cdot\nabla u^\varepsilon_{app} + u^\varepsilon_{app}\cdot\nabla w^\varepsilon \qquad\qquad & \\
+w^\varepsilon + (\eta_B+\beta y)(w^\varepsilon)^\perp + \nabla q^\varepsilon - \Delta w^\varepsilon = \tilde{R}^\varepsilon \quad & \text{in } \Omega\cup\Omega^\varepsilon_w\cup\Omega^\varepsilon_e, \\
\nabla\cdot w^\varepsilon = 0 \quad & \text{in } \Omega\cup\Omega^\varepsilon_w\cup\Omega^\varepsilon_e, \\
[w^\varepsilon]_{|\Sigma_w\cup\Sigma_e} = 0 \quad & \text{at } \Sigma_w\cup\Sigma_e, \\
\left[\frac{\partial w^\varepsilon}{\partial n} - q^\varepsilon\vec{n}\right]_{|\Sigma_w\cup\Sigma_e} = \tilde{\sigma}^\varepsilon \quad & \text{at } \Sigma_w\cup\Sigma_e, \\
w^\varepsilon = 0 \quad & \text{at } \Gamma^\varepsilon_w\cup\Gamma^\varepsilon_e
\end{aligned} \tag{93}$$

with

$$\|\tilde{R}^\varepsilon\|_{L_t^\infty(H^{-1}(\Omega^\varepsilon))} = O(\varepsilon), \quad \|\tilde{\sigma}^\varepsilon\|_{L_t^\infty\left(H^{1/2}(\Sigma_w\cup\Sigma_e)\right)} = O(\varepsilon).$$

Final estimates

The final estimates are very close to those of [12]. We get

$$\begin{aligned}
\partial_t\|w^\varepsilon(t,\cdot)\|^2_{L^2} + \|w^\varepsilon(t,\cdot)\|^2_{L^2} + \|\nabla w^\varepsilon(t,\cdot)\|^2_{L^2} \le \int_\sigma \tilde{\sigma}^\varepsilon\cdot w^\varepsilon(t,\cdot) + \int_{\Omega^\varepsilon} R^\varepsilon\cdot w^\varepsilon(t,\cdot) & \\
+ \int_{\Omega^\varepsilon}\left(w^\varepsilon\cdot\nabla u^\varepsilon_{app}\right)\cdot w^\varepsilon = I_1 + I_2 + I_3. &
\end{aligned} \tag{94}$$

- I_1 satisfies

$$\begin{aligned}
|I_1| \le\; & \|\tilde{\sigma}^\varepsilon\|_{L_t^\infty\left(H^{-1/2}(\Omega^\varepsilon)\right)}\,\|w^\varepsilon(t,\cdot)\|_{H^{1/2}(\Omega^\varepsilon)} \\
\le\; & C_1\,\varepsilon\,\|\nabla w^\varepsilon(t,\cdot)\|_{L^2(\Omega^\varepsilon)} \\
\le\; & C_1^2\varepsilon \;+\; \varepsilon\,\|\nabla w^\varepsilon(t,\cdot)\|^2_{L^2(\Omega^\varepsilon)}.
\end{aligned} \tag{95}$$

- Similarly, we get

$$|I_2| \leq \; C_2 \varepsilon \; + \; \varepsilon \; \|\nabla w^\varepsilon(t,\cdot)\|^2_{L^2}.$$

- I_3 is the most difficult to control, because the boundary layer part of u^ε_{app} has strong gradient. Indeed,

$$\nabla u^\varepsilon_{app} = \varepsilon^{-2} \, \nabla_w(y) \left(u_w \left(t, y, \frac{y}{\varepsilon}, \frac{x - \chi_w(y)}{\varepsilon} \right) \right) \; + \; \text{higher order terms in } \varepsilon,$$

where $u_w = \nabla_w(y) \Psi_w$. The worst term is then

$$J^\varepsilon = \varepsilon^{-2} \int_{\Omega^\varepsilon} \nabla_w(y) \left(u_w \left(t, y, \frac{y}{\varepsilon}, \frac{x - \chi_w(y)}{\varepsilon} \right) \right).$$

Proceeding *exactly* as in [12, page 1495], one obtains

$$\text{For } \; \|\phi\|_\infty \; \text{ small enough, } \; |J_\varepsilon| \; \leq \; \frac{1}{2} \, \|\nabla w^\varepsilon(t,\cdot)\|^2_{L^2(\Omega^\varepsilon)}.$$

which alllows to absorb this term into the one coming from viscosity, and to conclude thanks to a Gronwall lemma.

4 Rough topography

4.1 Formal Derivation

We try to build an approximate solution to system (2) in the following natural form:

$$\psi^\zeta_{app} = \psi^0(t,x,y) + \zeta \psi^1(t,x,y,\frac{x}{\zeta},\frac{y}{\zeta}) + \zeta^2 \psi^2(t,x,y,\frac{x}{\zeta},\frac{y}{\zeta}) + \dots \tag{96}$$

where ψ^0 describes the large scale circulation and the ψ^i's, $i \geq 1$ are correctors taking into account the effect of the rough topography, through the variables X, Y.

At order ζ^{-1}, we get the following equation on ψ^1:

$$\begin{aligned} \partial_t \Delta_X \psi^1 \; + \; \tilde{u}^1 \cdot \nabla_X \left(\Delta_X \psi^1 \right) \; + \; u^0 \cdot \nabla_X \left(\Delta_X \psi^1 \right) \; + \; \tilde{u}^1 \cdot \nabla_X \eta_B \; + \; r \Delta_X \psi^1 & \\ - \; \nu (\Delta_X)^2 \psi^1 \; + \; u^0 \cdot \nabla_X \eta_B = 0, & \quad (97) \end{aligned}$$

where $\tilde{u}^1 = \nabla^\perp_X \psi^1$, and $u^0 = \nabla^\perp_x \psi^0$. This last equation is of type (18), setting $U := u^0$ and $\widetilde{\psi} := \psi^1$.

We obtain the evolution equation on ψ^0, by averaging in X, Y the equation at order ζ^0. Note that the nonlinearity in (2) can be written ($u = \nabla^\perp \psi$):

$$\zeta \; u \cdot \nabla \Delta \psi = \zeta \; \Big(\partial_y (\operatorname{div} (u_1 \otimes u)) - \partial_x (\operatorname{div} (u_2 \otimes u)) \Big).$$

Thus, the largest nonlinear term with non-zero average appears at $O(\zeta)$ only. We get

$$\begin{aligned} \partial_t \Delta_x \psi^0 \; + \; \beta \partial_x \psi^0 \; + \; u^0 \cdot \nabla_x \overline{\eta_B} \; + \; r \Delta_x \psi^0 \; + & \\ \overline{\nabla_X \eta_B \cdot \nabla^\perp_x \psi^1 + \nabla_x \eta_B \cdot \nabla^\perp_X \psi^1} = \beta \; \operatorname{curl} \tau, & \end{aligned}$$

where the bar stands for the average in the rough variable X. Then,

$$\begin{aligned}
\overline{\nabla_{\mathrm{X}}\eta_B \cdot \nabla_{\mathrm{x}}^{\perp}\tilde{\psi}} &= \int_{X,Y} -\partial_X \eta_B \partial_y \tilde{\psi} \;+\; \partial_Y \eta_B \partial_x \tilde{\psi} \\
&= \int_{X,Y} \eta_B \partial_y \partial_X \tilde{\psi} \;-\; \eta_B \partial_x \partial_Y \tilde{\psi} \\
&= \partial_x \left[\int_{X,Y} -\eta_B \partial_Y \tilde{\psi} \right] \;+\; \partial_y \left[\int_{X,Y} \eta_B \partial_X \tilde{\psi} \right] \\
&\quad + \int_{X,Y} \partial_x \eta_B \partial_Y \tilde{\psi} \;-\; \int_{X,Y} \partial_y \eta_B \partial_X \tilde{\psi} \\
&= \mathrm{curl}_{\mathrm{x}} \left(- \int_{X,Y} \eta_B \nabla_{\mathrm{X}} \tilde{\psi} \right) \;-\; \overline{\nabla_{\mathrm{x}}\eta_B \nabla_{\mathrm{X}}^{\perp}\tilde{\psi}},
\end{aligned}$$

so that we recover the limit system (19).

4.2 The auxiliary systems

The aim of this section is to show that the systems formally derived above are well-posed.

4.2.1 Resolution of (18)

In system (18), variables x, y are simply parameters, so that this system is very close to classical geostrophic equations. A simple Galerkin scheme leads to the existence and uniqueness of the solution

$$\widetilde{\psi} \in L^\infty\left((0,T)\times\Omega; H^1(Q)\right) \cap L^2\left(0,T; L^\infty(\Omega; H^2(Q)\right), \quad \int_{\mathbb{T}^2} \widetilde{\psi}(t,\cdot) \;=0.$$

4.2.2 Dissipativity of $\mathcal{F}$

We now prove proposition 2.4. Let $U \in \left(L^\infty\left((0,T)\times\Omega\right)\right)^2$, and $\tilde{\psi}$ solution of (18). The dissipativity of $\mathcal{F}$ is equivalent to the inequality:

$$\int_0^t \left(- \int_{X,Y} \eta_B \nabla_{\mathrm{X}} \tilde{\psi} \right) \cdot U \;\geq\; 0. \tag{98}$$

An integration by parts gives

$$\left(- \int_{X,Y} \eta_B \nabla_{\mathrm{X}} \tilde{\psi} \right) \cdot U = \int_{X,Y} \tilde{\psi}\left(\nabla_{\mathrm{X}} \eta_B \;\cdot\; U\right) \tag{99}$$

Now, we get from (18) that

$$\begin{aligned}
\int_0^t \int_{X,Y} \tilde{\psi}\left(\nabla_{\mathrm{X}} \eta_B \;\cdot\; U\right) = \int_0^t \Bigg(& r \int_{X,Y} |\nabla_{\mathrm{X}} \tilde{\psi}|^2 \;+\; \nu \int_{X,Y} |\Delta_{\mathrm{X}} \psi|^2 \;+\; \frac{\partial_t}{2} \int_{X,Y} |\nabla_{\mathrm{X}} \tilde{\psi}|^2 \\
& + \int_{X,Y} (\tilde{u}\cdot\nabla_{\mathrm{X}}\widetilde{\psi})\Delta_{\mathrm{X}}\widetilde{\psi} \;+\; \int_{X,Y} (U\cdot\nabla_{\mathrm{X}}\widetilde{\psi})\Delta_{\mathrm{X}}\widetilde{\psi} + \int_{X,Y} (-\tilde{u}\cdot\nabla_{\mathrm{X}}\eta_B)\widetilde{\psi} \Bigg) \\
& = \int_0^t \left(\sum_{i=1}^6 I_i \right).
\end{aligned} \tag{100}$$

• As $\tilde{\psi}(0,\cdot) \equiv 0$, $\int_0^t I_3 = \dfrac{1}{2}|\nabla_{\mathrm{X}}\tilde{\psi}(t,\cdot)|^2 \geq 0$.

• We do not detail the treatment of I_4, I_5, I_6. Integrations by parts lead to $I_4 = I_5 = I_6 = 0$ using that U does not depend on X and $\widetilde{u} = \nabla_{\mathrm{X}}^{\perp}\widetilde{\psi}$.

Thus, we obtain inequality (98), which ends the proof.

4.2.3 Resolution of (19)

System (19) can be written

$$\mathrm{curl}_{\mathrm{x}}\Big(\partial_t u^0 + \nabla_{\mathrm{x}} p^0 - (\overline{\eta_B} + \beta y)(u^0)^{\perp} + r u^0 + \mathcal{F}(u^0)\Big) = 0, \tag{101}$$

for any pressure p^0, so that it is a kind of "semilinear Euler equation". Therefore, as we do not manage to control $\|\,\mathrm{curl}_{\mathrm{x}}\,\mathcal{F}(u^0)\|_\infty$, it seems difficult to prove the existence of global solutions. Hence, we look for existence of local in time strong solutions. The only difficult point is to check that $\mathcal{F}$ behaves well with respect to functions in the space $\big(L^\infty(0,T;H^m(\mathbb{T}^2))\big)^2$, $m > 1 = \frac{d}{2}$. Once we have appropriate bounds on $\mathcal{F}$, it is then routine to conclude through standard Galerkin or iterative schemes (see for instance [14] on classical Euler equations).

Let us fix $M > 0$, and set $B_T^\infty(0,M)$ the ball of radius M in $(L^\infty(0,T\times\mathbb{T}^2))^2$. We prove

Proposition 4.1 *Let $m > 1$. There exists $T > 0$, such that $\mathcal{F}$ sends $B_T^\infty(0,M) \cap \big(L^\infty(0,T;H^m(\mathbb{T}^2))\big)^2$ to $(L^\infty(0,T;H^m(\mathbb{T}^2)))^2$, and is uniformly Lipschitzian on bounded subsets.*

Remark: Note that we have to look at a solution u^0 in $(L^\infty((0,T)\times\Omega))^2$ in order to be able to pass to the limit in the nonlinear terms $\widetilde{u}^1\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\psi^1) + u^0\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\psi^1)$ which are in the equation satisfied by ψ^1.

Remark: If $u_0 = \nabla_{\mathrm{x}}^{\perp}\psi_0$ belongs to $\big(H^m\left(\mathbb{T}^2\right)\big)^2$, $m > 1$, and if we set $M > \|u_0\|_\infty$, the proposition 4.1 above allows us to build a solution in short time to (19), with u_0 as initial data.

Proof of proposition 4.1 : This proof relies on two lemmas. The first one is a generalization of the classical Schauder's lemma about the behaviour of $F(u)$ for $u \in H^m$ and F a smooth function (cf [16]).

Lemma 4.2 *Let $\mathcal{F}$ a $\mathcal{C}^\infty$ function from $\{U \in (L^\infty(0,T))^2,\quad \|U\|_\infty < M\}$ to $(L^\infty(0,T))^2$, such that $\mathcal{F}$ and its derivatives send bounded sets to bounded sets. Then $\mathcal{F}$ sends $B_T^\infty(0,M) \cap \big(L^\infty(0,T;H^m(\mathbb{T}^2))\big)^2$ to $(L^\infty(0,T;H^m(\mathbb{T}^2)))^2$, and is uniformly Lipschitzian on bounded subsets.*

The proof of this lemma is a straightforward adaptation of the proof of Schauder's lemma [16]. For the sake of brevity, we skip it and refer to [16] for all necessary details.

Let $U \in (L^\infty(0,T))^2$. We define $\mathcal{G}(U)$ as the solution $\Psi = \Psi(t,X,Y)$ of

$$\begin{cases} \partial_t\Delta_{\mathrm{X}}\Psi \,+\, u\cdot\nabla_{\mathrm{X}}\left(\Delta_{\mathrm{X}}\Psi\right) \,+\, U\cdot\nabla_{\mathrm{X}}\left(\Delta_{\mathrm{X}}\Psi\right) \,+\, u\cdot\nabla_{\mathrm{X}}\eta_B \,+\, r\Delta_{\mathrm{X}}\Psi \\ \qquad\qquad -\,\nu\left(\Delta_{\mathrm{X}}\right)^2\Psi \,+\, U\cdot\nabla_{\mathrm{X}}\eta_B = 0, \\ \qquad\qquad u = \nabla_{\mathrm{X}}^{\perp}\Psi = (-\partial_Y\Psi, \partial_X\Psi). \end{cases} \tag{102}$$

As already mentioned in the proof of theorem 2.3, such a system has a unique weak solution Ψ in the space

$$E := L^\infty\left(0,T;H^1(Q)\right)\cap L^2\left(0,T;H^2(Q)\right).$$

so that $\mathcal{G}$ is well defined as a function from $(L^\infty(0,T))^2$ to E. The link between $\mathcal{G}$ and our functional $\mathcal{F}$ is made through the relation

$$\mathcal{F}(u^0)(t,x,y) = -\int_{X,Y} \eta_B\ \nabla_X\Big(\mathcal{G}\left(u^0(\cdot,x,y)\right)\Big)(t,X,Y)\ dX\ dY \tag{103}$$

Proposition 4.1 is then a direct consequence of lemma 4.2 and of the following

Lemma 4.3 *There exists $T>0$ such that $\mathcal{G}$ is a $\mathcal{C}^\infty$ function from $\{u\in(L^\infty(0,T))^2,\quad \|u\|_\infty < M\}$ to E. Moreover, $\mathcal{G}$ and its derivatives send bounded sets to bounded sets.*

Remark: This lemma can be seen as a result of smooth dependence on parameters for 2D Navier-Stokes equations.

Proof of Lemma 4.3: We divide this proof into two parts

FIRST PART: We introduce an auxiliary function. Let $T>0$. We define

$$\mathcal{H} : (L^\infty(0,T))^2\times E\times E' \mapsto E,$$

by: $\mathcal{H}(U,\alpha,f)$ is the solution Ψ of

$$\begin{cases}\partial_t\Delta_{\rm X}\Psi\ +\ U\cdot\nabla_{\rm X}\left(\Delta_{\rm X}\Psi\ \right)\ +\ u\cdot\nabla_{\rm X}\left(\Delta_{\rm X}\alpha\right)\\ \qquad\qquad +\ u\cdot\nabla_{\rm X}\eta_B\ +\ r\Delta_{\rm X}\Psi -\ \nu\left(\Delta_{\rm X}\right)^2\Psi = f,\\ u = \nabla_{\rm X}^\perp\Psi = (-\partial_Y\Psi,\partial_X\Psi).\end{cases} \tag{104}$$

We show that $\mathcal{H}$ is a smooth function and that each of its derivatives sends bounded sets to bounded sets.

• Let $t>0$. Multiplying by $\mathbf{1}_{[0,t]}\Psi$ and integrating both over Q and from 0 to t leads to the estimate

$$\frac{\|\nabla_{\rm X}\Psi(t,\cdot)\|^2_{L^2_{\rm X}}}{2} + r\int_0^t \|\nabla\Psi(s,\cdot)\|^2_{L^2_{\rm X}} + \nu\int_0^t \|\Delta_{\rm X}\Psi(s,\cdot)\|^2_{L^2_{\rm X}} = -\langle\, f,\mathbf{1}_{[0,t]}\Psi\,\rangle_{\langle E',E\rangle}. \tag{105}$$

Hence,

$$\frac{\|\nabla_{\rm X}\Psi(t,\cdot)\|^2_{L^2_{\rm X}}}{2} + r\int_0^t \|\nabla_{\rm X}\Psi(s,\cdot)\|^2_{L^2_{\rm X}} + \nu\int_0^t \|\Delta_{\rm X}\Psi(s,\cdot)\|^2_{L^2_{\rm X}} \leq$$
$$C_1\ \|f\|_{E'}\ \left(\left(\int_0^t \|\Delta_{\rm X}\Psi(t,\cdot)\|^2_{L^2_{\rm X}}\right)^{1/2} + \left(\sup_{0\leq s\leq t}\|\nabla_{\rm X}\Psi(s,\cdot)\|_{L^2_{\rm X}}\right)\right) \tag{106}$$

Thus

$$\frac{\|\nabla_{\mathrm{X}}\Psi(t,\cdot)\|^2_{L^2_{\mathrm{X}}}}{2} + r\int_0^t \|\nabla\Psi(s,\cdot)\|^2_{L^2_{\mathrm{X}}} + \frac{\nu}{2}\int_0^t \|\Delta_{\mathrm{X}}\Psi(s,\cdot)\|^2_{L^2_{\mathrm{X}}} \leq$$
$$C_2\,\|f\|^2_{E'} \;+\; C_1\,\|f\|_{E'}\left(\sup_{0\leq s\leq t}\|\nabla_{\mathrm{X}}\Psi(s,\cdot)\|_{L^2_{\mathrm{X}}}\right) \quad (107)$$

Applying last inequality for t from 0 to T yields

$$\begin{aligned}\sup_{0\leq t\leq T}\frac{\|\nabla_{\mathrm{X}}\Psi(t,\cdot)\|^2_{L^2_{\mathrm{X}}}}{2} &\leq C_2\,\|f\|^2_{E'} \;+\; C_1\,\|f\|_{E'}\left(\sup_{0\leq t\leq T}\|\nabla_{\mathrm{X}}\Psi(t,\cdot)\|_{L^2_{\mathrm{X}}}\right)\\ &\leq C_2\,\|f\|^2_{E'} \;+\; C_3\,\|f\|^2_{E'} + \frac{1}{4}\sup_{0\leq t\leq T}\|\nabla_{\mathrm{X}}\Psi(t,\cdot)\|^2_{L^2_{\mathrm{X}}}\end{aligned} \quad (108)$$

Finally, we obtain

$$\|\mathcal{H}(U,\alpha,f)\|^2_E \leq C\,\|f\|^2_{E'} \quad (109)$$

where $\|\Psi\|^2_E = \|\Psi\|^2_{L^\infty(0,T;H^1(Q))} + \|\Psi\|^2_{L^2(0,T;H^2(Q))}$.

• We may now show that $\mathcal{H}$ is a smooth function. For two elements (U,α,f) and $(\tilde{U},\tilde{\alpha},\tilde{f})$ in the domain of $\mathcal{H}$, we set

$$\Psi = \mathcal{H}(U,\alpha,f), \quad \widetilde{\Psi} = \mathcal{H}(\tilde{U},\tilde{\alpha},\tilde{f}), \quad \chi = \widetilde{\Psi} - \Psi \quad (110)$$
$$V = \tilde{U} - U, \quad b = \tilde{\alpha} - \alpha, \quad g = \tilde{f} - f.$$

It is then straightforward to see that $\chi = \mathcal{H}(\tilde{U},\alpha,F)$, where

$$F = -V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi) - \nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b) + g.$$

Note that both $V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi)$ and $\nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)$ belong to E'. First, for all $\phi\in E$, we have

$$\begin{aligned}\left|\int_0^T\int_Q V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi)\,\phi\right| &= \left|\int_0^T\int_Q V\cdot\nabla_{\mathrm{X}}\phi\,(\Delta_{\mathrm{X}}\Psi)\right|\\ &\leq \|V\|_{L^\infty(0,T)}\,\|\nabla\phi\|_{L^2(0,T\times\Omega)}\,\|(\Delta_{\mathrm{X}}\Psi)\|_{L^2(0,T\times\Omega)}\\ &\leq C\|V\|_{L^\infty(0,T)}\,\|\Psi\|_E\,\|\phi\|_E.\end{aligned}$$

The other term is treated in the following way:

$$\begin{aligned}\left|\int_0^T\int_Q \nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)\,\phi\right| &= \left|\int_0^T\int_Q \nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}\phi\,(\Delta_{\mathrm{X}}b)\right|\\ &\leq \|\Delta_{\mathrm{X}}b\|_{L^2(0,T\times\Omega)}\,\|\nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}\phi\|_{L^2(0,T\times\Omega)}.\end{aligned}$$

Besides,

$$\begin{aligned}\|\nabla^\perp_{\mathrm{X}}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}\phi\|^2_{L^2(0,T\times\Omega)} &\leq \int_0^T\int_Q |\nabla_{\mathrm{X}}\widetilde{\Psi}|^2\,|\nabla_{\mathrm{X}}\phi|^2\\ &\leq \int_0^T \|\nabla_{\mathrm{X}}\widetilde{\Psi}(t,\cdot)\|^2_{L^4}\|\nabla_{\mathrm{X}}\phi(t,\cdot)\|^2_{L^4}.\end{aligned}$$

The Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
\|\nabla_{\mathrm{X}}^{\perp}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}\phi\|^2_{L^2(0,T\times\Omega)} &\le \int_0^T \|\nabla_{\mathrm{X}}\widetilde{\Psi}(t,\cdot)\|_{L^2}\,\|\Delta_{\mathrm{X}}\widetilde{\Psi}(t,\cdot)\|_{L^2}\,\|\nabla_{\mathrm{X}}\phi(t,\cdot)\|_{L^2}\,\|\Delta_{\mathrm{X}}\phi(t,\cdot)\|_{L^2} \\
&\le \|\nabla_{\mathrm{X}}\widetilde{\Psi}\|_{L^\infty(0,T;L^2)}\,\|\Delta_{\mathrm{X}}\widetilde{\Psi}\|_{L^2(0,T;L^2)}\,\|\nabla_{\mathrm{X}}\phi\|_{L^\infty(0,T;L^2)}\,\|\Delta_{\mathrm{X}}\phi\|_{L^2(0,T;L^2)}.
\end{aligned}$$

Thus, we get

$$\left|\int_0^T\int_Q \nabla_{\mathrm{X}}^{\perp}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)\,\phi\right| \le \|\widetilde{\Psi}\|_E\,\|b\|_E\,\|\phi\|_E. \tag{111}$$

Finally,

$$\|V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi)\|_{E'} \le \|V\|_{L^\infty(0,T)}\,\|\Psi\|_E, \quad \|\nabla_{\mathrm{X}}^{\perp}\widetilde{\Psi}\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)\|_{E'} \le \|\widetilde{\Psi}\|_E\,\|b\|_E. \tag{112}$$

Using estimates (109), (112), we deduce

$$\|\mathcal{H}(\tilde{U},\tilde{\alpha},\tilde{f})-\mathcal{H}(U,\alpha,f)\|^2 \le C\left(\|V\|^2+\|b\|^2+\|g\|^2\right), \tag{113}$$

where $\|\cdot\|$ stands for the various appropriate norms. Thus, function $\mathcal{H}$ is lipschitzian and sends bounded sets to bounded sets.

One checks easily that we may also write

$$\chi = \mathcal{H}(U,\alpha,F_1)+\mathcal{H}(U,\alpha,F_2) \tag{114}$$

where

$$\begin{aligned}
F_1 &= -\nabla^{\perp}\Psi\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)-V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi)+g, \\
F_2 &= -\nabla^{\perp}\chi\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)-V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\chi).
\end{aligned}$$

Note that we have used the linearity of $\mathcal{H}$ in its last variable. Thanks to estimates (109) and (113) we have

$$\|\mathcal{H}(U,\alpha,F_2)\| = O(\|V\|^2+\|b\|^2+\|g\|^2).$$

We have thus proved that $\mathcal{H}$ is differentiable and formula

$$d\mathcal{H}\big[(U,\alpha,f)\big](V,b,g) = \mathcal{H}(U,\alpha,F_1).$$

If we introduce the (smooth) bilinear application Q and linear application L respectively defined by:

$$\begin{aligned}
Q\left(\Psi,(V,b,g)\right) &= -\nabla^{\perp}\Psi\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}b)-V\cdot\nabla_{\mathrm{X}}(\Delta_{\mathrm{X}}\Psi), \\
L\left(V,b,g\right) &= g,
\end{aligned}$$

we have the relation

$$d\mathcal{H}\big[(U,\alpha,f)\big](\cdot) = \mathcal{H}\Big(U,\alpha,Q\left(\mathcal{H}(U,\alpha,f),\cdot\right)+L\left(\cdot\right)\Big). \tag{115}$$

Thanks to this last relation, one can use a bootstrap argument, show that $\mathcal{H}$ is smooth, and that each of its derivatives sends bounded sets to bounded sets.

SECOND PART: We link $\mathcal{G}$ to $\mathcal{H}$. We first note the identity

$$\mathcal{G}(U) = \mathcal{H}(U, \mathcal{G}(U), -U \cdot \nabla_{\mathrm{X}} \eta_B).$$

Thus, to prove that $\mathcal{G}(U)$ is smooth (for T small enough), and that all its derivatives send bounded sets to bounded sets, it is enough to prove, for all U, the invertibility of the derivative of the function

$$\mathcal{K} : \alpha \longrightarrow \alpha \ - \ \mathcal{H}(U, \alpha, -U \cdot \nabla_{\mathrm{X}} \eta_B).$$

Using (115) leads to

$$d\mathcal{K}[\alpha](\cdot) = Id \ - \ \mathcal{H}(U, \alpha, B(\Psi, \cdot))$$

where

$$B(\Psi, b) = -\nabla^{\perp}\Psi \cdot \nabla_{\mathrm{X}}(\Delta_{\mathrm{X}} b), \quad \Psi = \mathcal{H}(U, \alpha, -U \cdot \nabla_{\mathrm{X}} \eta_B).$$

Bounds (109) and (112) yields

$$|||\mathcal{H}(U, \alpha, B(\Psi, \cdot))||| \leq \|\Psi\|_E.$$

In the same spirit as (109), one can prove easily that

$$\|\Psi\|_E^2 \leq C \ |U|^2 \|\eta_B\|^2_{L^2(Q)} \ T,$$

so that

$$|||\mathcal{H}(U, \alpha, B(\Psi, \cdot))||| \leq \ \sqrt{C} M \ \|\eta_B\|_{L^2(Q)} \ \sqrt{T}$$

Thus, if $\sqrt{C} M \|\eta_B\|_{L^2(Q)} \sqrt{T} \ < \ 1$, then $d\mathcal{K}[\alpha]$ is invertible, which ends the proof of the proposition.

4.3 Convergence results

As usual, on the basis of the previous analysis, one can construct an approximate solution $u^{\zeta}_{app} \ = \ \nabla^{\perp}_{\mathrm{x}} \psi^{\zeta}_{app}$, where ψ^{ζ}_{app} is given by (96). Theorem 2.6 then follows from an energy estimate on $v = u^{\zeta} - u^{\zeta}_{app}$. Note that

$$u^{\zeta}_{app} \approx \nabla^{\perp}_{\mathrm{x}} \psi^0 + \nabla_{\mathrm{X}} \Psi^1, \quad \nabla_{\mathrm{x}} u^{\zeta}_{app} \approx \frac{1}{\zeta} \nabla^2_{\mathrm{X}} \Psi^1,$$

so that we have the following bound on the nonlinear term

$$\|\zeta v \cdot \nabla_{\mathrm{x}} u^{\zeta}_{app}(t, \cdot)\|_{L^2_{\mathrm{x}}} \leq C \|v\|_{L^2_{\mathrm{x}}}.$$

All other terms are easy to handle, so that we do not give further details.

Acknowledgments. This work has been partially supported by the GDR "Amplitude Equations and Qualitative Properties" (GDR CNRS 2103: EAQP) and by the IDOPT project in Grenoble.

References

[1] Barcilon, V., Constantin, P., and Titi, E. S. Existence of solutions to the Stommel-Charney model of the Gulf Stream. *SIAM J. Math. Anal. 19*, 6 (1988), 1355–1364.

[2] Blayo, E., and Verron, J. The no slip condition and separation of western boundary currents. *Journal of Physical Oceanography 26* (1995), 1938–1951.

[3] Bougeault, P., and Sadourny, R. *Dynamique de l'atmosphère et de l'océan*. Editions de l'Ecole Polytechnique, 2001.

[4] Bresch, D., and Colin, T. Some remarks on the derivation of the Sverdrup relation. *J. Math. Fluid Mech. 4*, 2 (2002), 95–108.

[5] Bresch, D., and Desjardins, B. Existence of global weak solutions for 2d viscous shallow water equations and convergence to the quasi-geostrophic model. To appear *Comm. Math. Phys.*, 2003.

[6] Bresch, D., Guillen-Gonzales, F., and Rodríguez-Bellido, M. A corrector for the sverdrup solution for a domain with islands. To appear *Applicable Anal.*, 2003.

[7] Colin, T. Remarks on a homogeneous model of ocean circulation. *Asymptotic Anal. 12*, 2 (1996), 153–168.

[8] Colin, T. Modèles stratifiés en mécanique des fluides géophysiques. *Ann. Math. Blaise-Pascal 2* (2002), 229–243.

[9] Constantin, P., and Wu, J. Behavior of solutions of 2D quasi-geostrophic equations. *SIAM J. Math. Anal. 30*, 5 (1999), 937–948 (electronic).

[10] Desjardins, B., and Grenier, E. On the homogeneous model of wind-driven ocean circulation. *SIAM J. Appl. Math. 60*, 1 (2000), 43–60 (electronic).

[11] Galdi, G. P. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, vol. 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[12] Gérard-Varet, D. Highly rotating fluids in rough domains. *J. Math. Pures Appl. 82* (2003), 1453–1498.

[13] Jäger, W., and Mikelić, A. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations 170*, 1 (2001), 96–122.

[14] Marchioro, C., and Pulvirenti, M. *Mathematical Theory of Incompressible Nonviscous Fluids*. Springer Verlag, 1994.

[15] Pedlosky, J. *Geophysical Fluid Dynamics*. Springer Verlag, 1979.

[16] Rauch, J., and Keel, M. Lectures on geometric optics. In *Hyperbolic equations and frequency interactions (Park City, UT, 1995)*, vol. 5 of *IAS/Park City Math. Ser.* Amer. Math. Soc., Providence, RI, 1999, pp. 383–466.

[17] Vanneste, J. Enhanced dissipation for quasi-geostrophic motion over small-scale topography. *J. Fluid. Mech. 407* (2000), 105–122.

[18] Vanneste, J. Nonlinear dynamics over rough topography: barotropic and stratified quasi-geostrophic theory. *J. Fluid. Mech. 474* (2003), 299–318.